



INTRODUCTORY GEOMETRY.

H. S. MacLEAN.

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INTRODUCTORY GEOMETRY.

BY

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PREFACE.

The object of this manual is to present a course of elementary geometry adapted to the needs of Public schools.

Part I is intended to furnish sufficient work for two years' study of the subject. Although essentially introductory in its character, yet it takes the pupil far enough to give him a good grasp of the fundamental principles of the science. The exercises at the beginning of each chapter are designed to lead the pupil to investigate for himself, as well as to acquaint him in a practical way with certain geometrical facts. These exercises are merely suggestive, not exhaustive. Definitions, axioms, and postulates are introduced as they are needed. Demonstrative work is not begun until the pupil has first gained a definite knowledge of the principles upon which it is based. The exercises in demonstrative geometry are carefully selected. By their position, and also by the type in which they are printed, they are easily distinguished from the practical exercises.

Part II consists of the first book of Euclid's *Elements*. In this much stress is laid on the study of geometry as a training in logical thinking. Many notes and exercises are given.

In the preparation of the book I have consulted a large number of standard works. From these I received much assistance, which I cheerfully acknowledge. It is impossible

for me to specify here what I owe to each. I have also received many suggestions from friends during the progress of the work. I am under special obligations to Professor Cochrane, of Wesley College, for giving me access to his large mathematical library, and for valuable criticisms; also, to Principal Schofield, of the Winnipeg Collegiate Institute, for assistance in reading some of the proof-sheets, and for practical suggestions.

Any corrections of errors, or criticisms tending to make the book more useful, will be thankfully received.

H. S. M.

Winnipeg, September 1, 1899.

PART I.

INTRODUCTORY COURSE.

GEOMETRY.

CHAPTER I.

INTRODUCTION.

Exercise I.

1. Point out three objects that are very different from one another in shape.

2. Point out three objects that resemble one another in shape.

3. Make a list of the names of objects which resemble in shape a football. A chalk-box.

4. Describe as fully as you can the shape of the objects lying on the table, viz. :

- (i) A cricket-ball.
- (ii) A box.
- (iii) A flat ruler.
- (iv) A new lead pencil.

5. Is it necessary to say anything about the material of which an object is composed in describing its shape? Give reasons for answer.

Exercise II.

1. Arrange according to size the objects lying on the table. Why do you regard this one as the largest?

2. Which is the largest article of furniture in the school-room? Why do you think so?

3. Compare these three pebbles as to size. Test by displacement of water.

4. Show how the difference in size of these tumblers (interior) may be represented.

5. Compare the size of the interior of a pint measure with that of a quart measure.

6. When do we say that one object is larger than another?

7. This box is completely filled with clay. Compare the size of the moulded clay with that of the interior of the box. Give reason for answer.

8. The room which any object takes up is usually spoken of as the *space* it occupies. Which of these blocks occupies the greater space? Test.

9. When do we say that one object is of the same size as another?

10. Place in a tumbler a quantity of water which is equal in size to this stone.

Exercise III.

1. Point out two objects of the same shape which are different in size.

2. Illustrate the fact that objects of the same size may be different in shape.

3. Find two objects which are of the same size and shape. Has the kind of material of which the objects are composed to be considered here? Why?

4. Find two objects which are exactly alike. How do you know that they are different objects?

5. How do we distinguish between different objects which are made of the same kind of material, and are of the same size and shape? Illustrate.

6. Place this block on one corner of the table. Describe as fully as you can its shape, size, and position.

7. Move the block in No. 6 to another position on the table. Does it now occupy the *same* space as before? Does it occupy the same *extent* of space? Is it of the same *shape* as before?

8. State what you have learnt about the *shape*, *size* and *position* of objects.

Exercise IV.

1. Examine a block of wood (rectangular). Describe its shape.

2. In what directions may this block be most conveniently measured? Name other objects which may be measured in a similar manner. Illustrate in the case of one of these.

3. A solid is said to have three dimensions—*length*, *breadth* and *thickness*. Point out the dimensions of a box, a book and a round ruler.

4. How many faces has this block? Point them out. How many are alike? Which is the greater, the left face or the front? The upper or the front? The lower or the front? Describe each face. Measure and draw each.

5. Which is the greatest of its dimensions? Which the least?

6. Find two other bodies which have each six faces. Find their dimensions. Describe one of these bodies.

7. Name six-sided bodies not found in the school-room.

Exercise V.

1. Examine the objects lying on the table. Point out the boundaries of each. [The boundaries of solids are called *surfaces*.]

2. How many boundaries has this block? Which is the largest? Which the smallest?

3. Point out and describe the boundaries of each of the following :

- (i) A box.
- (ii) A new lead pencil.
- (iii) A ball.

4. Place one block over another so that the edges of the surfaces in contact may coincide.

5. Point out the boundary between the blocks. Is it a part of either body? Has it any thickness? What are its dimensions ?

6. Cut an apple into two parts. Place the new surfaces in contact with each other. Does the boundary between the parts constitute any portion of either? What is its thickness? Give reasons for answer.

7. Point out the dimensions of the surfaces of this box.

8. Define *surface*.

9. Point out all the surfaces of this sheet of paper.

10. Can we think of any surface as being moved from one position to another without alteration of size or shape? Illustrate.

Exercise VI:

1. Examine this flat ruler. Point out the boundaries of its surfaces. Point out a common boundary of two of its surfaces. Has this boundary thickness? Give reason for answer.

2. Cut an apple into two parts. Point out the boundaries of its surfaces. Have these boundaries shape? Have they breadth or thickness? Give reasons for answer in each case.

3. Cut a sheet of smooth paper into two pieces and afterwards match the pieces carefully together on the desk. Point out all the surfaces of the paper.

4. What boundary (or boundaries) is common to both upper surfaces in No. 3? Is it a part of either surface? Has it any width?

5. Examine the boundaries of other surfaces. Have they breadth or thickness? In what respect may they be said to have size?

[The boundaries of surfaces are called *lines*.]

6. Is this fine thread a line? Give reasons for answer.

7. Show that a line must have shape, size and position. Can we think of a line as being moved from one position to another without alteration of size or shape? Illustrate.

8. When do we say that one line is equal to another? Illustrate.

Exercise VII.

1. Examine the surfaces of a block or box. Trace their boundaries.

2. Find where the ends of the lines bounding the different surfaces in No. 1 are situated.

3. Find the places at which the lines bounding the different surfaces in No. 1 meet.

4. Cut a sheet of paper (diagonally) into four pieces. Match the pieces carefully together. Show the lines bounding the upper surfaces of the four pieces. Show where lines meet. How many in each case? Show how to divide the sheet so that the ends of 8 lines may meet.

5. Represent on the blackboard two lines cutting each other. Does the point at which they cut each other occupy any part of the surface? State what has been learnt about the dimensions of lines. What can now be said about the size of a point?

6. What are points used for?

7. Can we think of a point as moving from one position to another? Illustrate.

Exercise VIII.

1. Define *solid*. Distinguish between a geometrical solid and a body.

2. Define *surface*. We say that a surface has size, shape and position. Illustrate by means of objects.

3. Is it possible without actually cutting the material of which a body is composed to think of its being divided into one or more parts by surfaces? Illustrate.

4. Define *line*.

5. Assuming that it is possible to think of a solid as being cut by surfaces in any way we choose, what inference can you make as to the position of lines on a surface? Illustrate.

6. Show by means of objects that a line has shape and size. Give illustrations of lines of different shapes and sizes.

7. Show that a line must have position.

8. Define *point*. From what you learnt about surfaces and lines can you tell why we may think of a point as being in any position on a surface? Illustrate your answer.

9. When do we say that one solid is equal to another? That one surface is equal to another? That one line is equal to another? Illustrate in each case.

EXPLANATIONS.

All objects which we can see, feel, handle, etc., have certain qualities such as shape, size, weight, colour, texture, temperature and position.

Geometry disregards all but three of the many qualities by means of which we distinguish objects from one another. These are *shape*, *size* and *position*. Geometry makes no distinction between a red-hot iron ball and a block of ice,

except in so far as these objects may differ in the three aforementioned characteristics.

I.—Space.

Any object which can be seen, felt, etc., must exist *somewhere*, that is, it must occupy some portion of space. Every object in a room occupies a portion of the space enclosed by the walls, floor, and ceiling of the room. The earth and the atmosphere by which it is surrounded occupy space and move rapidly through it. All objects in the universe exist in space. Space extends without limit in all directions, and is everywhere the same.

II.—Solids.

In order that an object may exist in space it must have *shape* and *size*. Without these characteristics it could not become known to us. It must lie in some particular place, or we could not distinguish it from other objects. When we say that two objects are exactly alike, we mean that they are alike in all respects but one, viz., that they occupy different

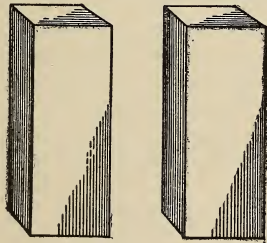


FIG. 1.

positions in space, as in Fig. 1. An object must, therefore, have *position*.

Again, we say that two bodies* are of the same shape and

* Any object occupying space is called a *body*. The space it occupies is called a *solid*. Thus, a bar of iron is a body, and the space it occupies is a solid.

size, if one of them will fill completely the space occupied by the other. In making this comparison we assume that *a body can be moved from one position to another without alteration of shape or size.*

Fill a tumbler with clay. The clay is of exactly the same shape and size as the interior of the vessel. Remove the clay and place the tumbler in its former position. Fill it with water. The water now occupies the same space previously occupied by the clay. The water in the tumbler and the moulded clay are bodies which are equal to each other.

When we disregard all the other qualities which a body may have and think of its *shape, size and position*, we have in mind a **geometrical solid**.

A solid is a limited portion of space. This definition implies that a solid takes up room, which fact is usually expressed by saying that a solid has three dimensions, viz., *length, breadth, and thickness*. The meaning of these terms will be made clearer hereafter.* But the definition also implies separation from other portions of space. Therefore a solid must have boundaries.

Suppose a body to be divided into any number of parts. Then each part must be a body because it occupies space. A crayon may be broken into small pieces, one or more of these pieces may be ground to a powder so fine that a single particle can scarcely be seen. Each particle is, however, a body composed of the same material as the original crayon, and the space which it occupies is a geometrical solid.

III.—Surfaces.

The boundaries of solids are called surfaces.

As a surface is that which separates portions of space which are in immediate contact with each other, a surface can have no thickness. It has two dimensions only, viz., *length* and

* The full meaning of the term *dimensions* will scarcely be comprehended until the generating of lines, surfaces and solids by movement in space is considered.

breadth. This implies that while a surface exists in space it takes up no part of it.

But as a surface forms the boundary of a solid which does occupy space, it must have *shape*, *size* and *position*. It can also be *moved freely in space* without alteration of shape or size.

Place the surfaces of two pieces of glass in contact with each other. We see that their common boundary includes no part of either piece of glass. It contains no material substance, therefore it has no thickness.

Pour water into a glass. The water and the glass have a common surface, and so have the water and the air above it. Evidently these surfaces have no thickness.

The same thing may be illustrated by placing one block upon another, as in Fig. 2.

A sheet of tissue paper is a body having two of its boundaries very close together. Gold leaf may be beaten out so thin as to require hundreds of thousands of layers of it to be placed together in order to form a body one inch in thickness. But the smallest conceivable portion of gold leaf occupies space, and must, therefore, be a body which has boundaries.

As a surface has no thickness, a solid, no matter how small, cannot be made up of surfaces.

Cut an apple into two or more pieces. Place the surfaces thus formed in contact with each other. Think of other surfaces that can be formed in a similar manner.

Thus, it is seen that we may think of a solid as being divided in any way we choose, that is, we may think of a surface as lying anywhere in the space occupied by the solid.

Again, as a solid can be moved from one position to another, it is evident that a surface which cuts it can be moved also.

It has been shown that surfaces have shape and size. They must, therefore, have boundaries.

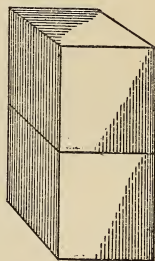


FIG. 2.

IV.—Lines.

The boundaries of surfaces are called lines.

Place edge to edge two panes of glass, or two sheets of paper. It is seen that the common boundary of their surfaces includes no part of either surface. As such boundary forms no part of the surface, it can have no breadth. Again, as a surface has no thickness, its boundary can have no thickness. But a surface has size, therefore that which forms the continuous limit of any portion of it must have size. The statement, “a line is that which has *length* but not breadth or thickness,” gives expression to these facts.

A line, like the surface it may be supposed to limit, has *shape, size and position*. It can, also, be *moved in space*.

Let a thin elastic band be stretched. Its cross-section diminishes as its length increases. Imagine that the stretching process continues until breadth and thickness are about to disappear. The band now represents a body having great length and almost no breadth or thickness. But as it still has breadth and thickness, however small, the space which it occupies is a solid. If attention be now fixed on the length of this solid, and if its breadth and thickness be disregarded, the resulting notion will be that of a geometrical line.

A line drawn on paper, as in Fig. 3, is not a geometrical line. Such a line must have breadth and thickness as well as length in order to be seen. It is therefore a solid, two of whose surfaces are very close together. One of these surfaces is exposed, the other is in contact with the paper. A line thus drawn is to be $\overline{A \hspace{10em} B}$ regarded merely as a *pictorial representation* of a geometrical line, that is, it is to be thought of as having one dimension only.

FIG. 3.

As a line has no breadth or thickness it will be seen that any number of lines, however great, placed close together cannot make up a surface. Such lines simply mark off narrow portions of surface.

Let a sheet of paper or other object be divided into any two parts. A new surface boundary or *line* has been made. Let other divisions of the surface be made. It is evident that we may think of a surface as

being divided in any way we choose. That is, we may think of a line as existing anywhere on a surface.

V.—Points.

It has been shown that a line may be thought of as lying in any position on a surface. Think of a line lying on the surface of a block. Suppose it be cut by another line on the same surface. As each line thought of has position, their intersection—the place where they meet—must have position. But as a line has neither breadth nor thickness one line cannot take up any part of the length of the other. Therefore the intersection of two lines has no shape or size. It is called a **point**.

A point denotes position only. *The intersections of lines are points.* A point can be *moved in space* as the surface on which it may be supposed to lie can be so moved.

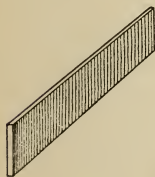


FIG. 4.

A line may be of limited length. The edges of a smooth board, as in Fig. 4, represent lines terminated at its corners. The extremities of a limited line must have position as the line has position. *The extremities of a line are points.*

Again, a line lying on a surface may be thought of as cut by lines at any points we may choose. It follows from this that we may take any point in a given line.

Describe on the blackboard a circle. Describe concentric circles each smaller than the preceding. At a certain stage a circle is represented by a small dot. Reduce this dot until it can scarcely be seen. Imagine that it is about to vanish. Now think of it as denoting position. The idea in mind will correspond to a **geometrical point**.

DEFINITIONS.

1. A limited portion of space is called a **solid**.
2. A boundary of a solid is called a **surface**.
3. A boundary of a surface is called a **line**.

4. The extremities of lines, or the intersections of lines are called **points**.

5. A solid, a surface, or a line is called a **geometrical magnitude**.

AXIOM.*

A solid, a surface, or a line can be moved in space without alteration of shape or size. A point can be moved from any position in space to any other position.

* An axiom is a self-evident truth.

CHAPTER II.

SOLIDS, SURFACES, LINES.

Exercise IX.

1. Examine a short piece of rough scantling. What are its dimensions? Describe its surfaces, edges, and corners.

2. Examine this model (rectangular parallelopiped). What are its dimensions? Describe the model.

3. Compare the surfaces of these solids. In what respects are they alike? In what respects are they different? Compare the surface of each with that of a window pane.

4. Name objects which resemble the model in shape.

5. Describe the form of the space which the model occupies.

6. Mould a parallelopiped. (The object removed from view.)

7. Examine this square prism. Show how it might be cut into two parts so that the space occupied by one part would be equal to that occupied by the other.

8. Show how this piece of scantling might be divided into four equal parts.

Exercise X.

1. Examine this model (cube).

2. Describe its boundaries. Compare them with those of the square prism.

3. Name other objects of the same form as this cube. Show wherein they resemble it. Show wherein they differ from it.

4. Think of the space which the model occupies. Describe that space. Is it a solid?

5. How does a cube differ from other six-sided bodies?

6. State in your own words the characteristics of a cube.

7. What is the distinction between the *form* and the *size* of an object? Illustrate by means of solids already examined.

8. Place one cube over another of equal size so that the edges of the surfaces in contact may coincide. Compare the space occupied by the body thus formed with that of one of the cubes.

9. Show how a cube may be divided into the least number of equal parts, each of which shall be a cube.

Exercise XI.

1. Examine an apple, an orange, a pear and a cricket-ball.

2. Tell in your own words what you can about the shape of each of the objects in No. 1.

3. Compare the objects in No. 1 as to form, pointing out differences and resemblances.

4. Examine this model (sphere). In what respects does its form differ from that of the other objects examined? Describe its form.

5. Select three other objects of the same form as this model. Think of the space they occupy. Arrange them in the order of size. Describe the form of the space each occupies.

6. Mould a sphere. Compare it with the model. Is the space the moulded object occupies a perfect sphere? Give reasons for answer.

7. State a practical method of determining whether a body which appears to be spherical has the form of a sphere or not.

8. Place in a vessel a quantity of water which will occupy the same extent of space as this model (sphere).

9. Give two methods of determining which is the greater of two spheres.

Exercise XII.

1. Examine a new lead pencil. In what respects does its form differ from objects already examined? How does it resemble them? Describe its form.

2. Compare the form of the lead pencil with that of this model (cylinder). Mould a cylinder. (Object removed from view.)

3. Describe the form of the space occupied by a cylinder.

4. Name manufactured articles which are cylindrical in form.

5. Make use of a practical method of determining whether this body is a cylinder or not.

6. Show how to cut a cylinder into two equal parts, so that each part may be a cylinder. Into four equal parts.

7. Place in a vessel a quantity of water which will occupy the same extent of space as this cylinder. Twice as much. Three times as much.

Exercise XIII.

1. Compare the Sphere, the Cube and the Cylinder as to the following :—

(a) Number of surfaces.

(b) Kinds of surfaces.

2. Show in a practical manner that there are two different kinds of smooth surfaces.

3. Name objects which have *curved* surfaces only.

4. Name objects which have *plane* surfaces only.

5. Name objects which have both curved and plane surfaces.

6. Apply a test for determining whether the surface of the blackboard is plane or curved.

7. Define *curved surface*. Define *plane surface*.

8. Mark off on the blackboard a surface equal to one of the surfaces of this piece of cardboard. Why do you say that these surfaces are equal? State the axiom on which the comparison of surfaces as to size is based.

9. Cut an apple into slices. Examine their flat surfaces and compare them as to extent.

10. Show how a plane surface must cut a sphere so as to divide it into two equal parts.

11. Show how to divide a hemisphere into four equal parts. How many dimensions has one of these parts? What are the dimensions of a sphere? Illustrate by cutting an apple.

Exercise XIV.

1. Examine a cylinder and a cube. Describe their surfaces.

2. Compare the surfaces of the cylinder with one another. Describe their boundaries.

3. Compare the line bounding the base of the cylinder with one of the lines bounding the upper surface of the table.

4. Examine this straight edge. What uses can we make of it? Illustrate.

5. Name objects having surfaces bounded by *curved* lines.

6. Name objects having surfaces bounded by *straight* lines.

7. Represent on the blackboard a surface enclosed by a curved line.

8. Represent a surface enclosed by straight lines.

9. Define *line*. Define *straight line*. Define *curved line*.

10. You wish to find out whether or not the edge of a flat ruler represents a straight line; what practical test will you apply? Illustrate.

11. If two straight lines coincide at their extremities, what can you say in regard to the position of the lines? Make

a statement which we can make use of as a test of the straightness of lines.

Exercise XV.

1. A surface is said to have only two dimensions. What are these? Explain fully.

2. Draw three surfaces of a square prism.

3. Draw the base of a cylinder.

4. Name objects which are bounded by curved surfaces only. By plane surfaces only. By both curved surfaces and plane surfaces.

5. What kind of line is formed by the intersection of two plane surfaces? By the intersection of a plane and a curved surface? Illustrate in each case.

6. Imagine that a plane surface cuts a sphere. Draw the line formed by the intersection of the surface of the sphere and the plane surface.

7. Produce the straight line AB both ways in a straight line. To what extent may its length be increased?

NOTE.—A pencil and flat ruler are used in drawing straight lines. When drawing a straight line through two points, place the pencil on one of the points and lay the ruler against it. Rotate the ruler about the pencil until its edge is at the second point. Hold the ruler in position and draw the line required. The line should pass *through* the points, not beside them.

Exercise XVI.

1. Find a body which can be rotated about a certain fixed* straight line without moving (as a whole) to a different position in space. Illustrate.

2. Find a body which can be rotated in any direction whatever about a certain fixed point without moving (as a whole)

* A solid, a surface, a line, or a point is said to be *fixed* when it is thought of as not subject to change of position while under consideration.

to a different position in space. Can a cylinder be so rotated? Illustrate.

3. Find two fixed points on the surface of a sphere about which it can be rotated without leaving (as a whole) the position in space which it occupies. Illustrate. Do the same in the case of a cylinder.

4. Fold a sheet of paper once. Suppose one of its surfaces to be divided into two parts by the crease. What does the crease represent? Lay one of the parts on a table and suppose its position fixed. Cause the other part to rotate about the crease. Is the shape or size of the moving surface affected by its motion? Give reasons for answer.

Fold again as before. Cut so that the boundaries of the surfaces in contact may coincide. Cause one of these surfaces to rotate about the crease until the whole sheet lies flat on the table. Compare the surfaces on opposite sides of the crease as to shape and size. State the axioms on which your conclusion is based.

5. Place this piece of thin cardboard so that it may lie flat on the surface of the blackboard. Outline the surface of the cardboard. Cause it to rotate about one edge which is straight until it again lies flat on the surface of the blackboard. Outline its surface in this position. What can you say as to the shape and size of the two figures thus drawn?

6. What practical applications can you make of the results arrived at in Nos. 4 and 5?

7. Take a fixed point on the surface of the blackboard. Draw through it a straight line. How many straight lines can be drawn through this fixed point?

8. Take two fixed points on the surface of the blackboard. Draw a straight line through both points. How many different straight lines can be drawn through both points? How many different curved lines?

9. How many fixed points on a straight line are necessary to determine the position of the line? How many points are necessary to determine the length of a straight line? Illustrate.

Exercise XVII.

1. Examine a parallelopiped. Show the surfaces, lines and points which it represents. Place it on a table and suppose it to be in a fixed position.

2. Suppose the point at one end of a line bounding the upper surface of the parallelopiped to move along this line until it coincides with a point at the other extremity. What has been traced by the moving point? Has its path length? Has it breadth? Has it thickness? Did the moving point change the direction of its course in passing from one position to the other? Give reasons for answer in each case.

3. Suppose the point to move from the same initial position and in the same direction as in No. 2. In this case let the motion be continuous without change of direction. Will the point ever return to its initial position? What will the path of the moving point represent? Whether is it limited or unlimited in the direction of motion? Give reasons.

4. Suppose a point to move along the same path as in No. 3, but in the opposite direction. Let the motion of the point be continuous without change of direction. What does the whole path described by the moving points now represent? Is it limited either way? Give reasons.

5. Define *straight line*. *Unlimited straight line*. *Finite straight line*.

6. Examine the base of a cylinder. Show the surfaces and lines which it represents. Place it on the table and suppose it to be fixed in position.

Suppose a point taken in the curved line bounding the upper base to move along this line. What does the path of the moving point represent? Suppose the motion to be continuous, what can you say about the position of the point?

7. Examine the upper surface of a table (rectangular). Show the lines and points represented. Place a stretched string so as to coincide with one of the straight lines bounding that surface. Move the string so that it may sweep over the whole surface of the table. Suppose the string to have neither breadth nor thickness, what does its path represent? How many dimensions has the path? Explain fully.

8. Suppose in No. 7 the motion of the line to be continuous without any change of direction, what will its path represent?

9. Show how a straight line must move through space to generate a plane surface. Is it possible for a straight line to move so as to generate a curved surface? Illustrate.

10. Show how a straight line must move in order (*a*) to cut off a cube from a square prism; (*b*) to cut a cube into pieces, each of which shall be a cube; (*c*) to cut a cylinder into two cylinders, (*d*) to generate the convex surface of a cylinder.

11. Place a small plate of sheet iron on the level surface of a quantity of fine damp sand, moulding clay or snow. Cause the plate to move downwards through the soft material. What inference do you make as to the path of a moving surface? How many dimensions has such path?

12. Show that it is possible for a line to move in space without generating a surface, and also, for a surface to move in space without generating a solid. Illustrate fully.

EXPLANATIONS.

It has been seen that solids, surfaces and lines have *shape*. The consideration of this characteristic leads to the classification of geometrical forms.

In this chapter the rectangular parallelopiped, the cylinder and the sphere are dealt with as representing types of solids, surfaces and lines.

I.—The Rectangular Parallelopiped.

A rectangular parallelopiped is bounded by six surfaces, called its sides or faces, all of which are plane. Its surfaces are bounded by straight lines, three of which meet at each of its corners. (See Fig. 5.)

That the parallelopiped is of three dimensions will appear obvious when the directions of the three lines meeting at one of its corners are considered.



FIG. 5.

[The square prism and the cube are particular cases of the rectangular parallelopiped, and they should be studied in connection with it.]

II.—The Cylinder.

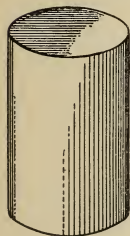


FIG. 6.

A cylinder* is bounded by three surfaces, one of which is curved and two are plane. The plane surfaces are called the *bases* of the cylinder; the other is called the *convex* surface. The bases are bounded by curved lines. (See Fig. 6.)

By placing the surface of a cube in contact with one of the bases of a cylinder so that a corner of the cube may coincide with the centre of the base, it may be readily seen that the cylinder is of three dimensions.

The same thing may be shown perhaps more clearly by quartering lengthwise any suitable cylindrical body.

If the centres of the bases of a cylinder be regarded as fixed points, then the cylinder may be rotated about the straight line joining them without changing its position (as a whole) in space.

* The term *cylinder* is here used in the same sense as *right cylinder*.

III.—The Sphere.

A sphere is bounded by one curved surface, every point on which is equally distant from a certain point within it, which is called the centre. (See Fig. 7.)



FIG. 7.

Cut an apple through the middle, and then quarter one of the pieces. By this means it may be easily shown that a sphere has three dimensions.

A sphere may be rotated about its centre in any direction whatever without changing its position in space. This may be illustrated by means of any small spherical body partially surrounded by fine damp sand. The ball and socket joint affords another illustration.

IV.—Classification of Magnitudes.

The three solids we have examined, viz., the parallelopiped, the cylinder, and the sphere, differ from one another in *shape*, and it is owing to this fact that they are given different names. Neither the size nor the position of solids is considered, when we divide them into classes and give to each class its name.

In like manner surfaces and lines are classified according to shape. Compare the curved surface of a globe with the plane surface of a smooth wall. Compare, also, the curved lines which bound the ends of a new lead pencil with the straight lines represented by the edges of a flat ruler.

V.—Motion in Space.

It will be observed that notions of geometrical surfaces, lines and points have been thus far related to, and derived from, the notion of geometrical solid. Naturally the beginner will assume that as a solid is limited, surfaces and lines must necessarily be limited also. That such is not the case may be made clear by considering two important facts stated in Chap. I.

- (i) *Space is unlimited.*
- (ii) *A solid, a surface, a line or a point may move freely in space.*

Suppose that a point moves in space. The path which it describes has only *one dimension*, viz., length. It is, therefore, a line. But the moving point may either return to its initial position as in tracing out the boundary of the base of a cylinder, or it may move onward through space never to return to any former position. Hence the path of a moving point—a line—may be confined to a limited portion of space, or it may be indefinitely extended.

Imagine a point to move in a straight line from P towards Q, as in Fig. 8. Space having no beginning or end, we may think of the point as moving onwards forever without deviating from its course. It will thus describe a straight line of unlimited length from P towards Q. We see, then, that it is possible for us to think of the limited line PQ as prolonged endlessly towards Q, as indicated by the arrow in the figure.



FIG. 8.

Similarly we may think of QP as prolonged endlessly towards P. (See postulate II).

Suppose the edge of a knife or a stretched string to represent a line. Cause it to pass through some soft substance, as moulding clay. Move a stretched string through the air so as to cut out a cube. We thus infer that a line moving through space usually generates a surface. We infer, also, that *a surface has two dimensions*; for it is generated by moving a line—a magnitude of one dimension—through space.

Fig. 9 represents a straight line which has described a plane surface in moving from its first position AB to a second position CD. The figure, shows, also, that we may think of a moving line as generating a surface which cuts a solid.

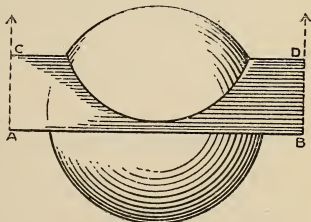


FIG. 9.

Again, a moving line may be confined to a limited portion of space, as in tracing out the curved surface of a right cylinder,

or it may move onward continuously through space, describing a surface of unlimited length. In the latter case if we suppose the moving line to be of unlimited length, it is evident that the surface described will be of unlimited length and breadth.

Suppose a thin piece of sheet iron to represent a surface. Cause it to be pressed into soft clay or other suitable substance.

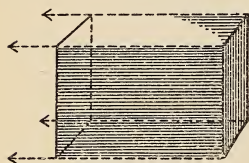


FIG. 10.

Move a square of paper through the air so as to describe a square prism. It is thus seen that a surface when moved through space usually generates a solid, as in Fig. 10. *A solid has three dimensions*, for it is described by moving a surface,

which has two dimensions, through space.

If we move a solid through space it is evident that a solid is again generated. Therefore, we conclude that *space has only three dimensions*.

DEFINITIONS.

6. A point has position, but it has no size.

A point is generally denoted by a capital letter. Its position is indicated by a small dot (\bullet) or cross (\times).

As a geometrical point has no size, such marks are to be regarded not as points, but as representations of points.

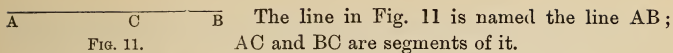
7. A line has position and length, but neither breadth nor thickness.

The extremities of a line are **points**.

The intersections of lines are **points**.

Any portion of a line is termed a **segment**.

A *geometrical* line is represented to the eye by a *visible* line.



8. A **straight line** is that which lies evenly between its extreme points.

This definition of straight line *expresses* the idea of straightness, but it does not, in reality, define anything. Euclid's assumption regarding the straight line conveys the full meaning of the definition. It is expressed as follows:

Two straight lines cannot enclose a space. This is equivalent to saying that two straight lines cannot coincide at two points without coinciding throughout their whole length. Thus, the lines shown in Fig. 12a, or Fig. 12b cannot both be straight lines.

It follows from this axiom that two points determine the position of a straight line. In other words, *one straight line, and only one, can pass through two points.*

Take any two points. Draw a straight line passing through both. Draw another straight line through the same points. Where must it lie?

How often can two straight lines cross each other? Illustrate.

Show how the foregoing axiom is applied practically, in testing the straightness of a line drawn on paper.

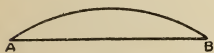


FIG. 12a.



FIG. 12b.

If the extremities of any part of a straight line are made to fall on any other part, these parts will, however placed, coincide throughout.

It has been seen that a terminated straight line may be produced indefinitely in both directions. A straight line thus produced is called an **unlimited straight line**.

A limited portion of an unlimited straight line is called a **finite straight line**. Thus if the straight line in Fig. 13 be unlimited, PQ will represent a finite portion of it.

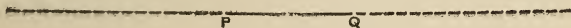


FIG. 13,

9. A line not straight, but made up of a series of straight lines, is called a **broken line**.

Thus, ABCD is a broken line. Its extreme points A and D are called *end-points*.

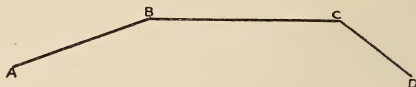


FIG. 14.

10. A **curved line** is a line of which no part is straight.

11. A **plane surface** is that in which any two points being taken, the straight line joining them lies wholly in that surface.

A plane surface is frequently called a **plane**.

In order to apply this definition practically in testing any surface, we place a straight-edge in contact with the surface first in one position, then in another, etc., until we are satisfied that we have determined whether or not the surface is a plane.

What part of the definition suggests the use of a straight-edge in testing a surface? Why is it necessary to place the straight-edge in different positions? Is it possible to make a perfect test of any surface in the manner indicated by the definition? Give reasons for answers.

12. A **curved surface** is a surface of which no part is a plane.

AXIOM.

Two straight lines cannot enclose a space.

POSTULATES.*

I. A straight line may be drawn from any one point to any other point.

*The postulates give us permission to draw only the very simplest figures. They serve merely as a starting-point in geometrical drawing. In *pure* geometry (elementary) we are limited to the use of the straight-edge, compasses, and pencil. (See postulate III.) No construction is allowable which cannot be effected by means of these instruments. In *practical* geometry, other instruments are found convenient; some of these are used freely in the earlier chapters of this book. As the student gains power in constructing geometrical figures he will gradually become independent of the assistance of all instruments other than the three already named, except in cases where standard units of measurement are given.

It is assumed that a straight line may exist anywhere in space. In geometrical representation this postulate permits the use of the pencil and straight-edge in drawing a straight line between any two points.

II. A terminated straight line may be produced to any length in a straight line.

It is here assumed that a straight line may be of unlimited length. Practically this postulate gives permission to use the pencil and straight-edge in extending the length of any given straight line.



CHAPTER III.

STRAIGHT LINES.

Exercise XVIII.

1. Draw a straight line equal to the edge of this cube. Why do you say that the straight line drawn is of the required length?

2. Draw on paper a straight line of exactly the same length as the straight line AB, here represented. $\overline{A \quad B}$. Why do you say that the line you have drawn is equal to AB?

3. Draw a straight line equal to the sum of the straight lines AB and CD. Equal to their difference.

$\overline{A \quad B} \quad \overline{C \quad D}$

4. Draw a straight line which is equal to the sum and difference of the straight lines PQ and RS, taken together.

$\overline{P \quad Q} \quad \overline{R \quad S}$

5. Compare the lengths of the straight lines AB and CD. Estimate and test.

$\overline{A \quad B} \quad \overline{C \quad D}$

6. Compare the lengths of the straight lines PQ and RS. Estimate and test.

$\overline{P \quad Q} \quad \overline{R \quad S}$

7. Taking the straight line AB as the standard of comparison, express the relation between the lengths of the straight lines AB and CD.

$\overline{A \quad B} \quad \overline{C \quad D}$

Express the same relation, taking CD as the standard of comparison.

8. On what axiom is the comparison of straight lines based? Show its application in each of the preceding cases.

9. Apply each of the following axioms to straight lines :

(a) Things which are equal to the same thing are equal to one another.

(b) Things which are doubles of the same thing are equal to one another.

(c) Things which are doubles of equal things are equal to one another.

Illustrate in each case.

10. The straight line PQ is of the same length as the straight line AB. The straight line RS is of the same length as the straight line AB. Compare PQ and RS as to length. Give reason for answer.

11. Taking your pencil as the unit of measurement, find the measure of the edge of the table. Express the relation thus found.

12. State what is meant by

(a) Quantity.

(b) Unit of measurement.

(c) Measure of quantity.

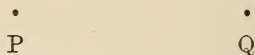
13. What is a standard unit? What are the advantages of using standard units in measuring quantities? Illustrate fully.

Exercise XIX.

1. Draw a straight line twice as long as your pencil. One-half as long.

2. Produce the straight line AB so that the part produced may be equal to AB. So that the part produced may be double of AB.

3. Find the distance between the points P and Q, and draw a line whose length is equal to that distance.



4. Take any two points A and B. Draw a number of different lines terminated at these points. Compare the lengths of the lines by placing on them a fine thread. Draw the line which is the shortest possible.

5. Find the distance between two points, P and Q, on the blackboard.

Along what line did you move the rule in measuring the distance? Why?

6. Take two points, A and B, fixed in position, on a plane surface. How many different straight lines can be drawn from A to B? How many different curved lines?

7. You are told that the fixed points A and B in a plane are the end-points of a curved line. Does that enable you to say where the line lies in the plane? Can you say what its length is? Illustrate.

If possible, draw a line from A to B, which must take a certain position and no other, so long as the points remain where they are. What kind of line is it? Can its length be changed (between the points) without changing the position of either of the points?

8. How many straight lines can be drawn through a point in a plane? How many straight lines can be drawn through eight points in a plane, each line joining two points? (No three points in the same straight line.) What is necessary to determine the position of a straight line in a plane?

Exercise XX.

1. Draw a straight line AB.
2. Draw a straight line CD, making it twice as long as AB in No. 1. (Use dividers for determining length.)
3. Draw a straight line EF, making it three times as long as CD in No. 2.
4. Draw a straight line GH equal to the sum of CD and EF in Nos. 2 and 3.
5. In the four preceding exercises, compare the length of AB with that of the other lines, using AB as the unit of comparison.

Compare length of AB, of CD and of EF with that of GH, the latter being the unit of comparison.

6. State the axioms which are applied in Nos. 2 and 4.
7. Estimate the length of the straight lines A, B, C, D, E and F on the blackboard. Test by measurement.
8. Mark on the blackboard the positions of points in a straight line which are 3 in., 6 in., 12 in., and 24 in. respectively from a given point P. Test.

9. Estimate in feet the length of the straight line PQ represented on the blackboard. Measure. Draw straight lines A, B, C, D, E and F, whose length shall be respectively $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{16}$ and $\frac{3}{4}$ of the given straight line PQ.

Compare A and B, A and C, C and D, C and E, D and F. The difference of A and B and the sum of C and D.

Draw to a scale of 1 in. per ft. straight lines representing the straight lines A, B, C, D, E and F.

10. Express the ratio between each of the following, using the quantity given first in order as the standard of comparison.

- (i) $2\frac{1}{2}$ in. and 2 ft. 6 in.
- (ii) 3 ft. 4 in. and 16 ft. 8 in.
- (iii) 5 ft. 6 in. and 16 ft. 6 in.
- (iv) 1 yd. 2 ft. 6 in. and 2 ft. 9 in.
- (v) 2 yds. 1 ft. 6 in. and 9 ft.
- (vi) 2 yds. 1 ft. 6 in. and 5 ft.

Exercise XXI.

1. By means of a straight stick (not measured) the distance between the points P and Q in a plane is found to be equal to the distance between the points P and R in the same plane. On what axiom is the conclusion based?

Suppose that by actual measurement with a foot-rule these distances are found in each case to be 2 ft. On what axiom or axioms do we base the assertion that the straight lines PQ and PR are equal to each other?

2. The straight lines PQ and ST are each equal to the straight line AB. The measure of the straight line LM is one half that of PQ and the measure of the straight line OX is three times that of ST. Compare LM and OX. State the axioms on which the conclusion is based.

3. Define *unit* of measurement. Show the importance of having units differing in magnitude.

A line is sometimes measured by *pacing* it. What is the objection to this mode of measurement?

The straight line PQ is $\frac{3}{4}$ as long as the straight line ST. What is the unit? Whether is this unit definite or indefinite?

4. What is meant by *drawing to scale*? Draw the following to a scale of $\frac{1}{2}$ inch per foot:

- (i) A straight line 3 ft. 9 in. long.
- (ii) " " " 4 ft. 8 in. "
- (iii) " " " 5 ft. 4 in. "
- (iv) " " " 9 ft. 3 in. "

5. Represent the following drawn to a scale of $\frac{1}{4}$ in. per ft.:

- (i) A straight line 6 ft. long.
- (ii) A straight line 10 ft. 6 in. long.
- (iii) A straight line 12 ft. 9 in. long.

6. State a practical method of finding the middle point of a string. On what axiom is this based? Illustrate.

7. Find by repeated trial with dividers:

- (i) The point at which the straight line AB is bisected.*

A B

- (ii) The points at which the straight line PQ is trisected.†

P Q

- (iii) A straight line whose length is one-twelfth that of the straight line RS.

R S

8. Measure a side of the school ground. Place a stake at each extremity and at the middle point of the side.

9. Represent on paper the line in No. 8, drawn to a scale of 1 inch per rod.

EXPLANATIONS.

I.—Comparison.

The fundamental truth upon which the comparison of geometrical magnitudes is based is stated by Euclid, thus: "*Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.*"

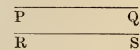
* To *bisect* a magnitude is to divide it into *two equal parts*. The point at which a finite straight line is bisected, is called the *point of bisection*.

† To *trisect* a magnitude is to divide it into *three equal parts*.

This axiom—frequently called the **principle of superposition**—implies :

- (i) That magnitudes may be compared quantitatively.
- (ii) That magnitudes may be moved in space without alteration of shape or size.

As applied to straight lines the axiom may be stated thus :
If one finite straight line be placed upon another so that their extremities coincide, the straight lines are equal.

Let PQ and RS be finite straight lines. Suppose PQ  to be shifted from its position so that the point P will fall on R, and that the straight line PQ will fall on the straight line RS. The point Q must coincide with some point on RS or RS produced. If the point Q falls on the point S, then according to the foregoing axiom PQ is equal to RS. FIG. 15.

A finite straight line is fully determined when *the two points at its extremities* are known, as these points determine both its length and position. An unlimited straight line is fully determined when *any two points in it* are known. Why?

II.—Distance.

As the *straight* line joining two points is the only line whose length and position are fully determined by the points, its length is regarded as the **distance** between the points.

A straight line is the shortest that can be drawn from one point to another.

This almost self-evident truth may be illustrated experimentally as follows : Drive two tacks into a wall, at a convenient distance apart. Fasten to one of them a cord which passes over the other. Pull the loose end of the cord, and the part of it between the tacks will become shorter and shorter until it is straight ; beyond this limit its length cannot be diminished. Give two other practical illustrations of this fact.

Although the truth of the foregoing statement is so evident as scarcely to require proof, yet it is not to be assumed as an axiom.

Geometers make it a rule not to assume anything that can be proved. For the present, however, we shall regard the mechanical proof given above as sufficient.

III.—Sum and Difference.

If any number of straight lines, AB, CD, EF, placed end to end, form the straight line XY, as in Fig. 16, then XY is said to be the **sum** of AB, CD and EF. Any other straight line whose end-points can be made to coincide with X and Y is equal to the sum of AB, CD and EF.

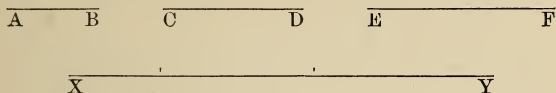


FIG. 16.

Also, AB is the **difference** between XY and the sum of CD and EF. Make corresponding statements regarding CD and EF. Give three illustrations of the *sums* of straight lines. Of the *differences* of two straight lines.

IV.—Measurement.

When one straight line AB is contained in another CD, an exact number of times, AB is said to be a measure of CD, and CD is said to be a multiple of AB.

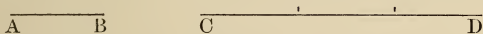


FIG. 17.

A straight line is measured by applying to it another line whose length is known in terms of a standard unit of length. For short lines the foot rule, the yard stick, and the ten-foot pole are generally used as measures or units; for long lines the chain and the tape are used. [A rod pole will be found convenient for field measurements in connection with school work.]



FIG. 18.

The instruments used in measuring distances and drawing straight lines on paper are the compasses (dividers), the ruler, the scale, the pencil and the drawing pen.

To measure a straight line. Place the points of a pair of dividers so that one is at each end of the line as in Fig. 18. The distance between the points is then measured by means of a scale, as in Fig. 19. When great accuracy is required, a diagonal scale is used.

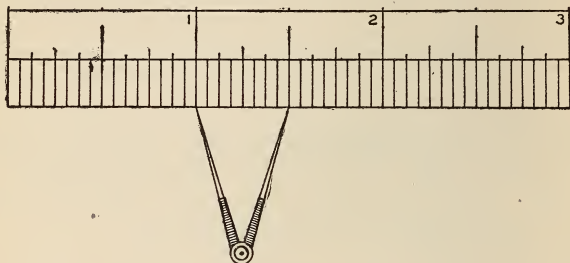


FIG. 19.

Such a scale, showing hundredths of an inch, is represented by the diagram given below (Fig. 20). The dots are at a distance of 2.64 inches apart. The use of the diagonal scale will be better understood; however, at a more advanced stage.

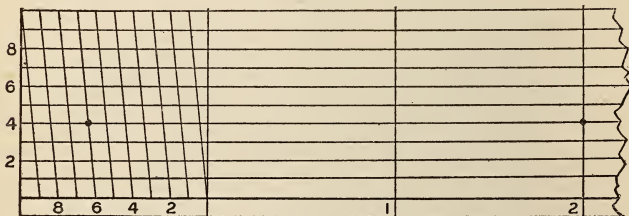


FIG. 20.

To set off a line of any given length. Open the dividers so that the points will indicate the given length on the scale, then carry the distance to the required position, as in Fig. 21.

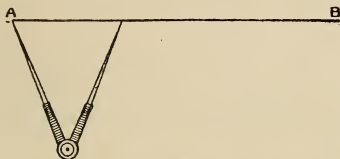


FIG. 21.

To draw a line of given length. Place the edge of a ruler in the required position of the line: at a point A indicating one of its extremities, place a point of the dividers—already set for the distance. The other point of the dividers when placed at the edge of the ruler will mark the other extremity B of the line, as in Fig. 22. The line is then drawn with a

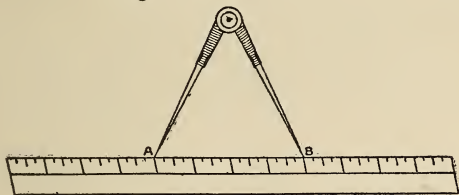


FIG. 22.

pencil or drawing pen. Lines thus drawn should always be tested as to accuracy of length.

A pencil used for drawing straight lines should have a fine wedge-shaped point.

V.—Ratio.

Draw any straight line AB unlimited towards B, as in Fig. 23. Take any point C in it. Set off CD, DE, and EF, each equal to AC. Then AD is twice as long as AC, AE three times as long, etc.

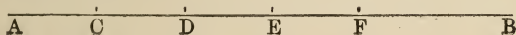


FIG. 23.

Thus we see that straight lines may bear to one another a relation which we express by the terms, *twice, three times, etc.* This relation is called **ratio**.

The terms, *one-half, one-third, three-fourths, etc.*, are also used to express ratio.

In placing the edge of a cube on paper and drawing a straight line corresponding to it in length, the axiom of superposition is applied ; for it is assumed that the line drawn and the line represented by the edge of the cube coincide.

Also, in drawing a straight line, say, twice as long as the edge of a cube, the same axiom is applied ; for the line may be regarded as made up of two parts, each of which coincides with the edge of the cube. Another assumption, however, must be made, viz., *that straight lines which are equal to the same straight line are equal to one another* ; hence, the equality of the two straight lines forming the parts of the whole straight line.

VI.—Drawing to Scale.

One practical application of the comparison of straight lines consists in representing on paper or other suitable material measurements of land, buildings, etc., by means of diagrams drawn accurately to scale. For example, let the side of a building 40 feet long be represented by a straight line 10 inches long. In this case the length of the line in the diagram will be $\frac{1}{4}$ of that of the line it represents.

The ratio of the shorter line to the longer is called the *reducing factor*. In the preceding example the reducing factor is, therefore, $\frac{1}{4}$. This ratio might also be expressed by saying that the diagram is drawn to a scale of $\frac{1}{4}$ inch per foot.

To construct a scale. Suppose the reducing factor to be $\frac{1}{12}$. Then every inch on the scale will represent 12 inches or 1 foot, and every twelfth of an inch will represent 1 inch.

Draw a straight line of unlimited length. Beginning at some point in it, set off a part one inch long, then another

part equal to the first, etc., until the whole length thus set off is sufficient to measure the longest line in the diagram to be made. Number the parts consecutively as in Fig. 24.

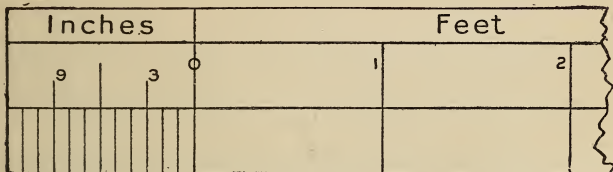


FIG. 24.

If inches are to be represented, measure off one inch, and divide it into twelve equal parts.* These should also be numbered at intervals, in order that the scale may be convenient for use.

Fig. 25 shows two different scales. In both, the figures indicate the number of feet. Determine the reducing factor in each case.

Draw a straight line 10 ft. 6 in. long to the three scales shown by Figs. 24 and 25. Account for the ratio which the three lines thus drawn bear to one another.

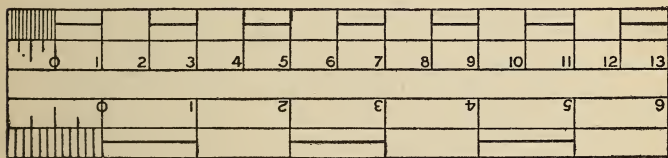


FIG. 25.

The scale to which any diagram should be drawn will depend on the purpose it is intended to serve. This is well illustrated by maps. Compare the scales to which different maps in the school-room are drawn. Account for differences.

* At this stage the student is not in a position to divide a straight line into any given number of equal parts. He can, however, do all that is required here by setting off, in succession, four lines, each $\frac{1}{4}$ inch long. The three subdivisions of these may be estimated with considerable accuracy.

DEFINITIONS.

13. A **magnitude** or **quantity** is that which can be increased, diminished, and measured. Lines, angles, surfaces and solids are geometrical magnitudes.

Strictly speaking, the foregoing is not a definition, as the terms *increased*, *diminished* and *measured* all involve the idea of magnitude or quantity. [*Magnitude* denotes a fundamental space conception; therefore, it cannot be defined by means of simpler terms.]

14. A number which expresses how many times any quantity contains another quantity of the same kind is said to be the **measure** of the former quantity in terms of the latter.

Thus when we say that a straight line 12 inches long contains a straight line 3 inches long four times, the number four is the measure of the 12-inch line in terms of the 3-inch line.

15. When one magnitude is compared with another of the same kind the quantitative relation which exists between them is called the **ratio** of the magnitudes.

Thus when we say that a straight line 12 inches long is four times as great as a straight line 3 inches long, the number four expresses the ratio between the lengths of the lines.

16. A **unit** is any quantity which is used in determining the measure of another quantity of the same kind.

Thus when we say that the straight line AB is three times as great as the straight line CD the latter is taken as the unit of comparison. But we may express the relation by saying that CD is one-third of AB, when we take AB as the unit of comparison.

17. A **standard unit** is a unit whose quantitative value is fixed by law or usage. For example, the yard is the standard unit of length in Canada, Great Britain and the United States.

A geometrical magnitude—line, angle, surface, or solid—may be defined in two ways.

- (i) By stating the ratio between it and some other magnitude of the same kind which is not measured. Thus we may say that one straight line is double of another without expressing the length of either in terms of any standard unit,

- (ii) By stating the ratio between it and another magnitude of the same kind which is itself either a standard unit, or defined in terms of a standard unit. Thus we may say that a straight line 6 yards long is six times as great as the standard unit 1 yard; or, that it is twice as great as a straight line 3 yards long.

In the first of these cases we are concerned with only the ratio of the magnitudes; in the second we take account of not only the ratio of the magnitudes, but also of their quantitative values as expressed in terms of a fixed unit.

Frequently geometrical magnitudes are such that they cannot be expressed exactly in terms of a common unit. The mode of treatment applicable to such magnitudes is beyond the scope of an elementary book.

18. The **distance** between two given points is the length of the straight line joining them.

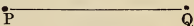
Thus, the distance between the points P and Q  is the length of the straight line PQ.

FIG. 26.

Points are said to be *equidistant* from a given point, when the straight lines drawn from them to the given point are equal.

AXIOMS.*

1. Things which are equal to the same thing, or to equal things, are equal to one another.
2. If equals be added to equals, the sums are equal.
3. If equals be taken from equals, the remainders are equal.
4. If equals be added to unequals, the sums are unequal in the same sense.
5. If equals be taken from unequals, the remainders are unequal in the same sense.
6. Things which are doubles of the same thing, or of equal things, are equal to one another.

* In stating the axioms here, and elsewhere, it is not the intention to acquaint the student with the truths which they express. This knowledge must be founded on *experience with things*.

7. Things which are halves of the same thing, or of equal things, are equal to one another.

A. Magnitudes can be moved freely in space, without alteration of shape, or size.

8. Magnitudes that can be made to coincide with one another, are equal.

9. The whole is greater than its part, and equal to all its parts, taken together.

10. Two straight lines cannot enclose a space.

An axiom is a simple statement, the truth of which is so obvious, as to admit of no doubt whatever.

The requirements of an axiom are as follows :

- (i) It must be *self-evident*.
- (ii) It must *not be based on any simpler truth*.
- (iii) It must *serve as a starting-point from which reasoning may proceed*.

The first seven axioms, and also the ninth, are applicable to all magnitudes. They are, therefore, called *general axioms*.

Axioms A, 8 and 10 are called *geometrical axioms*, because they can be applied to geometrical magnitudes only.

State each of the general axioms, and show how it is applied in the case of straight lines. Illustrate.

CHAPTER IV.

RELATIVE POSITION OF STRAIGHT LINES.

Exercise XXII.

1. Is it possible for two straight lines that do not lie in the same plane to cut each other? Illustrate by means of two flat rulers.

2. Define *unlimited straight line*. Is a straight line 1000 miles long unlimited?

3. Examine the blackboard. What kind of surface does it represent? What does each of its edges represent? Point out two of its edges which meet. Two which do not meet.

4. Examine a triangular prism. What kind of surfaces has it?

Point out the straight lines bounding its surfaces as follows:

(a) Those which do not meet.

(b) Those which meet.

5. Examine the straight lines drawn on the blackboard.

(a) Point out two which meet at a point.

(b) Point out two which would meet if produced. Produce them until they meet.

(c) Point out two which cannot meet.

6. Draw through the point P on the blackboard a straight line meeting the straight line AB. How many straight lines may be thus drawn through P?

7. Draw through the point Q on the blackboard a straight line which would not meet the straight line AB, however far both lines might be produced.

[Two straight lines in the same plane which do not meet, however far produced both ways, are said to be *parallel* to each other.]

8. Point out parallel edges of objects in the school-room. Edges which are not parallel.

9. Represent on paper

I. Two parallel straight lines.

II. Two straight lines which do not actually meet, but which would meet if produced.

III. Two straight lines which meet at a point.

Exercise XXIII.

1. A straight line AB is drawn on the blackboard. Hold a fine cord stretched along the surface of the blackboard so that

I. Its extremities may fall on AB. In what position does the cord lie? Why?

II. Its extremities may fall on opposite sides of AB. Give it different positions.

III. It may intersect AB at the point C. Give it different positions, if possible.

IV. A marked point on the cord may coincide with the point C. Give it different positions, if possible.

V. It may meet AB produced towards B.

VI. It may not meet AB, however much its length or that of AB may be increased.

Compare the position of the cord with that of AB in each of the foregoing. When a fixed point on the cord meets AB at only one point P, what positions may it take? Is there any position which it cannot take? State the conditions under which the cord (*a*) must fall on AB; (*b*) must intersect AB; (*c*) must be parallel to AB.

2. Through the point P draw a straight line parallel to the given straight line AB.
 3. Through the points P, Q, and R draw straight lines parallel to the straight line LM. Are all these lines parallel to one another? Test by means of instruments.
 4. Test the parallelism of the straight lines drawn on a sheet of paper.
 5. Draw through a point in AB any straight line which does not coincide with AB. Through a point in this line draw a straight line parallel to AB.
 6. Through the points P and Q in AB, draw two straight lines parallel to each other.
-

EXPLANATIONS.

I.—Relative Position.

All solids, surfaces, lines and points must lie somewhere in space, that is, they must have *position*. It is evident, however, that the position of any particular magnitude, or point, cannot be defined by reference to itself. Its position must be considered in relation to some other thing, whose position for the time being is supposed to be known. This is seen in the use we make of terms denoting position, as, *above*, *below*, *between*, etc.

A line has only three characteristics, viz., shape, length, and position. When we say a line is straight, we define its *shape*; when we state the ratio which it bears to some other line whose length is known, we define its *length*; when we tell where it lies, we define its *position*.

We have already learnt that the position of a straight line which passes through two given points is known. Suppose such a line to be drawn on the blackboard. Now let us think

of the position of another straight line, also represented on the blackboard. Are the lines near each other or far from each other? Do they cross each other or do they run alongside of each other without crossing? When we thus think of the position of one straight line in connection with that of another, we have in mind the **relative position** of the lines.

II.—Coincidence, Intersection and Parallelism.

Let AB be a straight line whose position in a plane is fixed. Let PQ be a straight line which is movable from any one position to any other in the plane.

As PQ is a movable straight line it may be given any position in the plane whatsoever.

I. Let PQ be placed so that the points P and Q may coincide with *two* points in AB as in Fig. 27. Then every point in PQ must fall on AB.

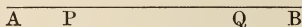


FIG. 27.

Two straight lines which have *two points in common* are **coincident** with each other; that is, they *lie in the same straight line*.

When a movable straight line is placed so as to fall upon another straight line, the two constitute one and the same straight line. When we speak of two straight lines in a plane we mean, unless otherwise stated, straight lines which do not coincide with each other.

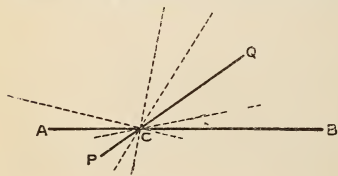


FIG. 28.

II. Let PQ be placed so as to have only one point C in common with AB, as in Fig. 28. It is evident that PQ may take any number of positions and yet pass through the point C. There is but one

position which it cannot take, viz., that of coincidence with AB; for then it would have more than one point in common with AB. In every position which PQ may take it *intersects* AB.

Draw a straight line AB on the blackboard. Let another straight line be represented by a fine cord stretched along the surface of the blackboard so as to meet AB at any point C. Cause the cord to rotate about C, keeping it stretched along the surface of the blackboard. It is obvious that the cord may take any number of different positions and at the same time meet AB at only one point C. Mark several of its positions by means of dotted lines. Notice how it is related to AB in each of these.

Suppose PQ or PQ produced towards P or Q to meet AB or AB produced towards A or B at one point, as in Figs. 29 and 30. In both of these positions PQ and AB are regarded as *intersecting* lines.

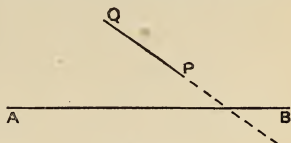


FIG. 29.

Two straight lines (produced if necessary) which have *only one point in common intersect* at that point, *forming an opening with each other*.

III. Let PQ be placed so as to have no point in common with AB. This will be the case if PQ be so situated in the plane as not to meet AB, however far PQ and AB may be produced both ways.

This relation may be made clearer by the following considerations :

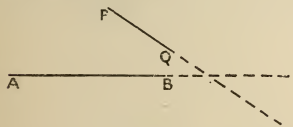


FIG. 30.

Suppose PQ to be coincident with AB. Then let PQ be moved to any other position in the plane in which it will not coincide with AB at any point.

PQ cannot take the position shown by Fig. 30 because PQ and AB approach each other towards Q and B, and, consequently, they meet when produced far enough. For a similar

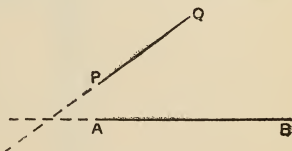


FIG. 31.

reason PQ cannot take the position shown by Fig. 31. It is

now evident that PQ must take some position such as that shown by Fig. 32.

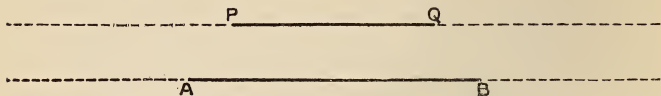


FIG. 32.

Two straight lines in a plane which have *no point in common*, that is, which do not meet however far they may be produced both ways, are said to be **parallel** to each other.

To draw through a given point a straight line parallel to a given straight line.

Let AB be the given straight line and P the given point.

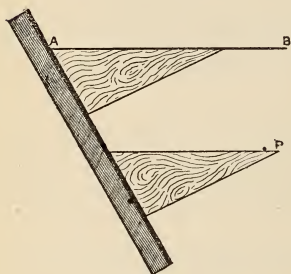


FIG. 33.

Place one edge of a set-square against the line AB and place a ruler against another edge of the set-square as in Fig. 33. Hold the ruler firmly in position and slide the set-square along the surface of the paper, keeping it in contact with the ruler, until the edge which coincided with the straight line AB comes

to the point P. This edge will indicate the position of the required straight line.

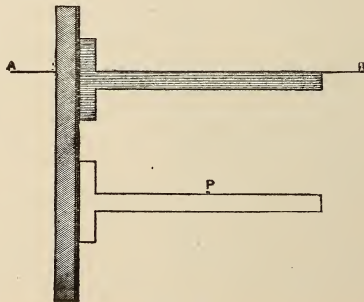


FIG. 34.

Instead of a set-square, or triangle, as it is frequently called, a **T**-square may be used. Fig. 34 shows how a **T**-square and ruler are placed in drawing parallel lines.



DEFINITION.

19. Two straight lines in a plane which do not meet, however far produced both ways, are **parallel** to each other.



CHAPTER V.

ANGLES.

Exercise XXIV.

1. Draw a straight line AB. From A draw any straight line AC as in Fig. 35.

[When two straight lines are drawn from a point they form an *angle*.]

2. Draw other straight lines AD, AE, etc., from the point A in No. 1. Compare the angles which these straight lines form with AB. On what does the size of an angle depend? Draw several angles differing in size.



FIG. 35.

3. What is meant by the *arms* of an angle? What is meant by the *vertex* of an angle?

4. At what hour do the hands of a clock point in the same direction from the pivot about which they turn? In opposite directions?

[Draw a straight line AB. Let the straight line AC rotate about the point A from the position of AB, as indicated by the arrow in Fig. 36. (Observe that the angle BAC increases as AC turns about A.) When AC coincides with BA produced the angle BAC is called a *straight angle*, and it is said to contain 180 degrees.]



FIG. 36.

5. From the point P draw in opposite directions the straight lines PA and PB. What is the measure of the angle made by them? Define *straight angle*.

6. From the point P, in the straight line AB, draw a straight line PC so that the angle APC may be equal to the

angle BPC. Apply a practical test. What is the measure of the angle made by the straight lines PA and PC?

7. Define *right angle*. Point out objects whose edges are at right angles to each other.

8. From the point P in the straight line LM draw a straight line PQ so that the angle QPM may be less than a right angle. Test with square. Compare the size of the angle QPL with that of a right angle.

9. Define *acute angle*. Define *obtuse angle*. Find illustrations of each.

Exercise XXV.

1. Examine this *protractor*. What do the figures on it indicate? What uses can you make of it?

2. Represent on a sheet of paper angles of which the measurements are as follows:—(a) 90° , (b) 45° , (c) 30° , (d) 15° , (e) 60° , (f) 75° , (g) 120° , (h) 135° . (Use protractor.)

3. When is one straight line perpendicular to another?

4. At the point C, in the straight line AB, erect a perpendicular. How many lines may be drawn through the point C perpendicular to AB in a plane? (Use set-square.)

5. Measure the angle PQR as in Fig. 37. Make an angle twice as great as PQR. Three times as great.

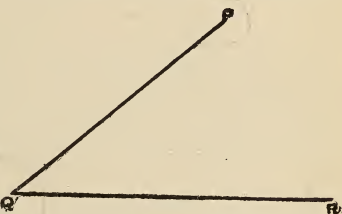


FIG. 37.

6. Fold a rectangular sheet of paper so that the angle formed by two creases may be an angle of 45° . Test by means of a protractor.

7. Draw an angle of 120° and bisect it. (Use protractor.) Test the equality of the parts by superposition.

8. Draw an angle ABC , as in Fig. 38. Cut out the figure. Use it in making an angle equal to ABC . Make an angle twice as great as ABC . Three times as great. One half as great.

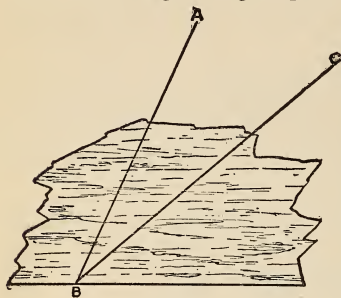


FIG. 38.

9. Draw, using protractor, two angles of 45° and 90° having a common vertex. Show by superposition that one angle is double of the other.

10. Draw two angles ABC and DEF . Make an angle equal to their sum. Equal to their difference.

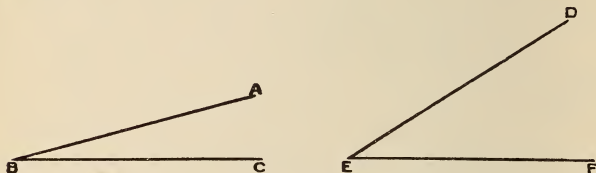


FIG. 39.

11. Two straight lines PA and PB 6 yds. and 8 yds. long, respectively, are drawn from the point P so as to form a right angle. Draw to a scale of $\frac{1}{2}$ in. per yard. Find by measurement of diagram the distance between the points A and B .

12. Draw to a scale of 2 in. per rod a straight line, 10 rods long, running due north from the point P , and locate the following :

I. A point due east of P at a distance of 5 rods.

II. A point at a distance of 5 rods due west of a point which is 10 rods from P and due north of it.

In what direction from the point first determined does the straight line joining the two points run? Find (approximately) its length. Where does it meet the line running north from P?

EXPLANATIONS.

In Chapter IV it was seen that when two straight lines meet at a point they form an opening with each other. In this chapter, we shall consider how the relative position of two straight lines may be defined in terms of the opening they make with each other, both lines being drawn from the same point.

I.—Plane Angle.

Let P and Q be two given points in a plane. From P draw a straight line which shall pass through the point Q.

Let Y be any point in the plane not in the straight line drawn from P *towards* Q (however far produced). From P draw a straight line which shall pass through Y. The

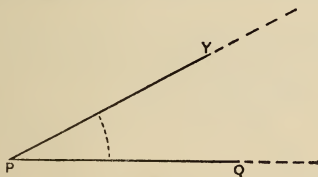


FIG. 40.

straight lines PQ and PY form with each other an opening, the size of which determines the relative position of the lines.

When two straight lines are thus drawn from a point, they make with each other, a **plane angle**.

Thus, in Fig. 39 the straight lines PQ and PY form the angle QPY of which the point P is the *vertex*. PQ and PY are called the *arms* (sometimes the *sides*) of the angle QPY.

II.—Size of Angles.

The size of an angle depends upon the opening formed by its arms, and not upon their length. If we regard the arms as two straight lines drawn in different directions from the vertex, then the size of the angle corresponds to the *difference in direction* of these lines.*

Fasten two threads to a point. Stretch the threads so that the opening between them may be very small. Increase it gradually until it becomes as large as possible. Tell how the threads are related to each other in different positions. Does the length of the threads affect the size of the opening formed by them, if not, what does affect it? Illustrate fully.

III.—Comparison of Angles.

As all angles contained by straight lines are plane angles, it is evident that such angles may be compared as to magnitude by the method of superposition.

Fasten two fine threads to any point and hold them stretched so as to form an angle. It is obvious that in every position they may take, a movable plane surface can be placed so that both threads will lie on it.

Let ABC and DEF be two angles whose magnitudes are to be compared with each other.

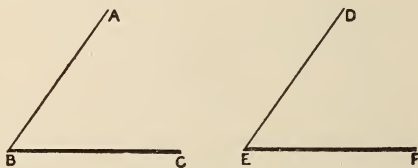


FIG. 41.

According to the assumption already made that a geome-

* If the meaning of the term *direction*, as applied to straight lines, is not fully understood, its use only serves to confuse the beginner. On the other hand, when its meaning is comprehended, it may be employed to advantage in developing the notion of angle.

trical magnitude may be moved from any one position in space to any other without alteration of shape or size, let the angle ABC be moved from its position and placed as follows :

- (i) So that it will lie in the same plane as the angle DEF .
- (ii) So that the point B will coincide with the point E .
- (iii) So that the straight line BC will fall on the straight line EF .

As the angles ABC and DEF are in the same plane ; as their angular points B and E are coincident ; and, as the straight line BC falls on the straight line EF , the magnitudes of the angles may be compared by considering the positions of the lines BA and ED . If BA falls between ED and EF , then must the angle ABC be *less* than the angle DEF . If BA falls on ED , then must the angle ABC be *equal* to the angle DEF . If BA falls on the side of ED remote from EF , then must the angle ABC be *greater* than the angle DEF .

It is thus seen that an angle is a quantity which may be compared with, and, consequently, measured by, another quantity of the same kind.

The quantitative aspect of angles is presented, more clearly, by the following method of comparison :—

Place a foot-rule with one edge PQ on a table as in Fig. 42. Open the rule gradually as indicated by the arrow. Notice the changes in

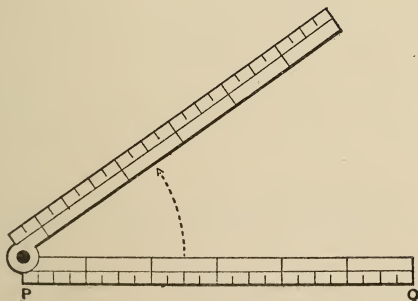


FIG. 42.

the opening between the inner edges of its parts as motion of one of them takes place. In what position do these edges represent coincident straight lines? In what position do they represent two straight lines drawn in opposite directions? In what position is their relation to each other the same as that of two adjacent edges of a sheet of foolscap? Open out the rule so as to make the angle between the inner edges as great as possible. Make the angle one half as great as possible.

From the preceding we can easily infer that when two straight lines are drawn from a point in a plane one of them may be regarded as stationary or fixed in position, and the other, as movable about the fixed point from which both are drawn.

Let P and Q be two given points in a plane. From P

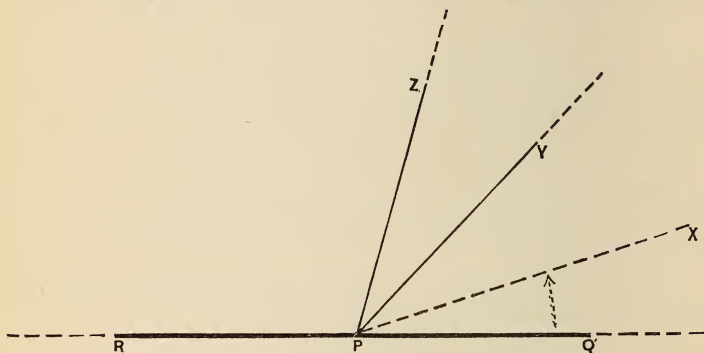


FIG. 43.

draw a straight line in the direction of the point Q . Produce PQ backwards, and let the straight line thus produced pass through the point R . PQ and PR are two straight lines drawn from the point P in opposite directions. From P draw two other straight lines passing through the points Y and Z , respectively.

Let the straight line PX be movable in the plane about the fixed point P . Suppose PX , in its initial position, to coincide with PQ . Let it rotate about P , as indicated by the arrow, until it coincides with PR . Its direction is now

exactly opposite to that of its initial position. It is obvious that in turning about P, PX would in one of its positions coincide with PY, and in another, with PZ. But, as the amount of rotation in passing from PQ to PZ is greater than in passing from PQ to PY, the angle QPZ is said to be greater than the angle QPY. Similarly, the angle QPR is said to be greater than the angle QPZ.

The size of any angle contained by two straight lines may therefore be determined by the *amount of rotation* (change of direction) in the plane of the angle, which a straight line must undergo about the point from which the two straight lines are drawn, in passing from the position of one of these lines to that of the other.

IV.—Units.

Angles are measured by comparing them with other angles of definite size, called **units**. (See Def. 16.)

The units of angular measurement, commonly used, are determined as follows :

(i) By drawing from a point in a given straight line another straight line, so that the adjacent angles thus formed may be equal, we get an angle of fixed magnitude which is called a **right angle**. (See Fig. 53.)

That all right angles are equal to one another is easily shown by the method of superposition. Euclid asserts the equality of right angles without proof.

When two straight lines are at right angles to one another, each is said to be **perpendicular** to the other.

(ii) By drawing from a point in a plane two straight lines in opposite directions, an angle of fixed magnitude, called a **straight angle**, is formed. (See Fig. 51.)

That all straight angles are equal to one another follows directly from the fact, that two straight lines coinciding at two points, must coincide

throughout their entire length. The equality of straight angles may therefore be proved by superposition.

(iii) By supposing a straight line to rotate, as in Fig. 44, in a plane about one of its extremities until it returns to its initial position, we get an angle of fixed magnitude which is



FIG. 44.

called an **angle of rotation** or **perigon**.^{*} If an angle of rotation be divided into two equal parts, each part must be a *straight angle*; if into four equal parts, then each part must be a *right angle*.

The right angle, straight angle, and angle of rotation, are formed by straight lines in certain limiting positions, only. This accounts for the fact that these angles are of definite size.

The only standard employed by Euclid is the right angle. (See Euclid's definition of *plane rectilineal angle*.)

For purposes of calculation and practical measurement, it is found convenient to divide an angle of rotation into 360 equal parts, called **degrees** ($^{\circ}$). A degree is divided into 60 equal parts, called **minutes** ($'$). The sixtieth part of a minute is called a **second** ($''$).[†]

V.—Measurement.

In measuring angles and in representing angles of given magnitude, an instrument called a protractor is generally used. It consists of a thin plate of brass or other suitable material cut in the form of a semicircle. It is graduated so as to show degrees. (See Fig. 45.)

^{*}The term "perigon" was invented by Halsted.

[†]The terms degree, minute, and second, are also used to denote divisions of the circumference of a circle.

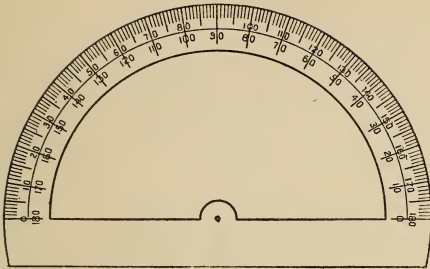


FIG. 45.

To measure an angle. Place the protractor so that its centre will coincide with the vertex, and its base will fall on one of the arms of the angle. The number corresponding to the radiating line falling on the other arm will indicate the measure of the angle in degrees.

For out-door work in measuring angles a simple instrument consisting of two thin pieces of wood, shaped like flat rulers and fastened together by a pin at their centres, will serve very well for school purposes.

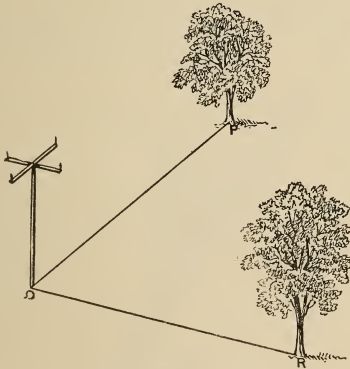


FIG. 46.

Vertical needles placed near the ends of the pieces of wood in the straight lines drawn on their upper surfaces through their common axis, are used for sighting. Fig. 46 shows the instrument set in position for

measuring the angle PQR, one of the lines of sight passing through the object at P and the other through the object at R. The angle at Q may be determined approximately, by measuring with a protractor the angle made by the sight lines marked on the instrument.

By placing the lines of sight at right angles to each other this contrivance may be used for erecting perpendiculars and finding the positions of offsets* to straight lines.

VI.—Drawing.

At a point in a given straight line to make an angle equal to a given angle.

Place the protractor as in measuring an angle. The radiating line whose number corresponds to the measure of the given angle in degrees, will indicate the position of the straight line to be drawn.

To draw a straight line perpendicular to a given straight line from a given point.

Let AB be the given straight line to which a perpendicular is to be drawn through the given point P.

Place the longest edge of a set-square against the straight line AB, as in Fig. 47. Place a ruler against a second edge

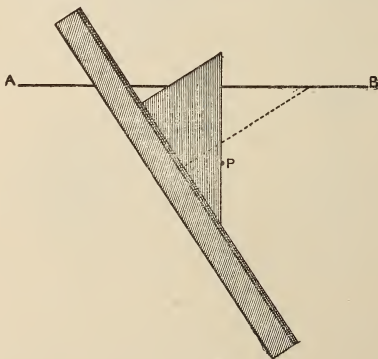


FIG. 47.

* Surveyors use the term *offset* to denote a short line measured perpendicularly to a longer one.

of the set-square. Holding the ruler in position, turn the set-square until its third side is in contact with the ruler. Slide the set-square along the ruler until its longest edge comes to the point P. This edge gives the position of the required straight line.

DEFINITIONS.

20. Any two straight lines drawn from a point, are said to make with each other, a **plane angle**.

The following is Euclid's definition :

A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

The two straight lines containing the angle are called the **arms** of the angle, and the point where they meet is called the **vertex** of the

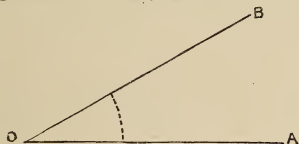


FIG. 48.

angle. Thus in Fig. 48 AOB or BOA is the angle ; OA and OB are its arms ; and the point O is the vertex. This angle may also be called the angle at O.

When several angles have a common vertex each angle is denoted by three letters ; but these letters must be placed so that the one at the

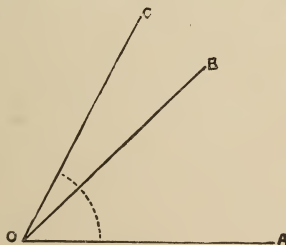


FIG. 49.

vertex will be between the other two. Thus in Fig. 49 three angles have a common vertex, viz., AOB or BOA , BOC or COB , and AOC or COA .

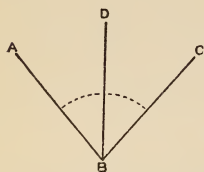


FIG. 50.

21. Two angles having a common vertex, and lying on opposite sides of a common arm, are called **adjacent angles**.

Thus ABD and CBD , in Fig. 50, are adjacent angles.

The angle ABC , being made up of the angles ABD and CBD , is said to be the **sum** of the angles ABD and CBD . The reason for this will be clearly seen if we suppose a straight line to rotate about B from the position of BC to that of BD , and then to that of BA .

Similarly, the angle ABD is said to be **difference** of the angles ABC and CBD .

To **bisect** an angle is to divide it into two equal angles.

A straight line which divides an angle into two equal angles is called the **bisector** of the angle.

22. A **STRAIGHT ANGLE** is an angle whose arms, being extended in opposite directions, are in the same straight line.



FIG. 51.

In Fig. 51 the point P is the vertex of the angle QPR , whose arms PQ and PR are in the same straight line.

23. An angle which is greater than a straight angle, but less than two straight angles, is called a **reflex angle**.

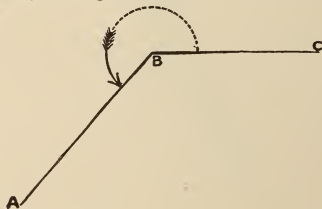


FIG. 52.

Suppose a straight line to rotate in a plane about the point B, as in Fig. 52, from the position of coincidence with BC to that of coincidence with BA, as indicated by the arrow. This line would turn through the reflex angle ABC.

24. When a straight line standing on another straight line makes the adjacent angles equal, each of these angles is called a **right angle**.

Thus if the straight line PQ in Fig. 53 stands on the straight line AB so as to make the angles PQA and PQB equal to one another, then each of the angles PQA and PQB is called a right angle. Hence, a right angle is half a straight angle; or, a straight angle is equal to two right angles.

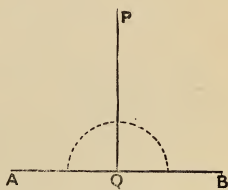


FIG. 53.

25. Two straight lines forming a right angle are said to be **perpendicular** to each other.

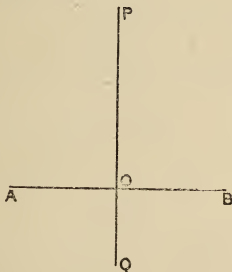


FIG. 54.

If a straight line PQ be at right angles to the straight line AB, intersecting it at O, PO is said to be the perpendicular from P on AB, and QO is said to be the perpendicular from Q on AB. Similarly AO and BO are perpendiculars on PQ from A and B. Also, each of the lines PQ and AB is said to be perpendicular to the other.

26. An angle which is greater than a right angle, but less than a straight angle, is called an **obtuse angle**.

In Fig. 55 the angle ABC is obtuse.

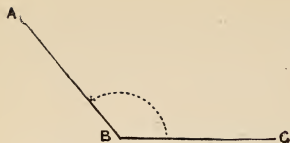


FIG. 55.

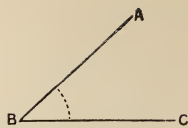


FIG. 56.

27. An angle which is less than a right angle is called an **acute angle**.

In Fig. 56 the angle ABC is acute.

28. When two straight lines cut one another, the opposite angles are called **vertically opposite angles**.

In Fig. 57 AOD and BOC are vertically opposite angles; also, the angles AOC and BOD.

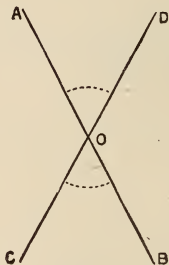


FIG. 57.

29. When the sum of two angles is equal to a straight angle, each of these angles is called the **supplement** of the other.

The angle ACD in Fig. 58 is the supplement of the angle BCD, also, the angle BCD is the supplement of the angle ACD.

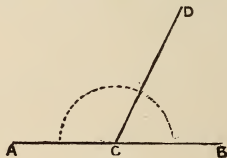


FIG. 58.

30. When the sum of two angles is equal to a right angle each of these angles is called the **complement** of the other.

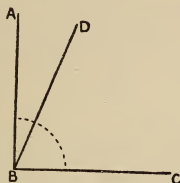


FIG. 59.

The angle DBC in Fig. 59 is the complement of the angle ABD ; also, the angle ABD is the complement of the angle DBC.

THEOREMS.

SECTION A—STRAIGHT LINES AND ANGLES.

A **theorem** is a statement of a geometrical truth which is to be demonstrated.

Example : ' If two straight lines cut one another, then the vertically opposite angles are equal.'

Every theorem consists of two parts :

- (i) The *hypothesis*, or *what is given*.
- (ii) The *conclusion*, or *what is to be proved*.

In the foregoing theorem the two parts are as follows :

Hypothesis—Two straight lines cut one another.

Conclusion—The vertically opposite angles are equal.

Whenever there is doubt as to essential parts of a theorem, it may be expressed in the form : ' If *A* is *B*, then *C* is *D*,' where *A* is *B* denotes the hypothesis, and *C* is *D* the conclusion.

I.—Demonstration.

To prove a theorem is to show by a process of reasoning that *the conclusion necessarily follows from the hypothesis*.

In proving a theorem the object is not merely to become convinced that it is true, but to show how it is connected

with, and follows from, certain definitions, axioms, or other truths already demonstrated.

Let AB be a given straight line, and let C and D be two points in it. If AC is equal to BD , prove that AD is equal to BC .

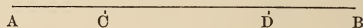


Fig. 60.

(i) We first ask what the given conditions are, for we cannot begin to reason until we have these clearly in mind. We find that AC and BD are equal parts of the straight line AB .

(ii) Our next question is what are we required to prove? We have to prove that AD must be equal to BC .

(iii) Referring to the diagram we see that AD is made up of AC and CD ; also, that BC is made up of BD and CD . We therefore conclude that as AC and BD are equal, AD and BC must be equal also.

(iv) Although we have seen that the conclusion necessarily follows from the hypothesis, we have not made a clear, well-arranged statement of the process of reasoning, such as a geometrical proof demands. To do this is our next task.

Commencing with the hypothesis we now arrange the whole in the form of an argument thus :

Given. That AC and BD are equal parts of the straight line AB .

To be proved. That AD must be equal to BC .

Proof. Because in the straight line AB the part AC is equal to the part BD ;

Given.

to each of these equals add the part CD ,

therefore, the sum of AC and CD is equal to the sum of BD and CD .

Ax.1.

That is, AD is equal to BC .

Q.E.D.

Q.E.D. (*quod erat demonstrandum*) is usually placed at the end of a theorem to denote that the truth of the theorem has been demonstrated.

A *corollary* is a truth which is easily deduced from a theorem.

1. A , B , C and D are four straight lines. Given that A is equal to C , and B equal to D , prove :

(i) That the sum of A and B is equal to the sum of C and D .

(ii) That the difference of A and B is equal to the difference of C and D .

2. In Fig. 61 the angle $\angle ABE$ is given equal to the angle $\angle CBD$. Prove that the angle $\angle ABD$ is equal to the angle $\angle CBE$.

3. The straight line AB is greater than the straight line CD . E and F are points in AB and CD respectively. If AE is equal to CF , prove that EB is greater than FD .

4. Illustrate axioms 1-7 as applied (a) to straight lines, (b) to angles.

5. Two straight lines which coincide for a part of their lengths will coincide throughout. Illustrate this axiom.*

6. State the *axiom of superposition*. What must be shown before we can say that two angles are equal? Illustrate. (See page 54.)

7. Define *straight angle*. Draw the straight angle $\angle KLM$. What does the definition tell you about the points K , L , and M ?

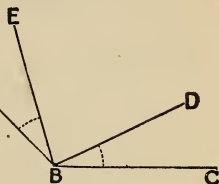


FIG. 61.

II.—Theorem 1.

All straight angles are equal.

Given. That the angles $\angle ABC$ and $\angle DEF$ are straight angles.

To be proved. That the angle $\angle ABC$ is equal to the angle $\angle DEF$.

Proof. Let the angle $\angle ABC$ be applied to the angle $\angle DEF$ so that the vertex B may be on the vertex E , and the arm BA may lie along the arm ED ;

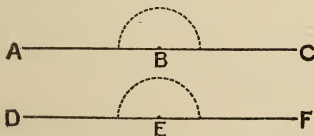


FIG. 62.

then because AC and DF are straight lines,

Def. 22.

therefore the arm BC must fall along the arm EF .

Ax. 10.

* This is not a new axiom; it is merely one form of Axiom 10. Another form is this: "Only one straight line can pass through two points."

Hence the angle ABC coincides with the angle DEF .

Therefore the angle ABC is equal to the angle DEF . *Ax.8.*

Q.E.D.

COROLLARIES :

1. *The supplements of equal angles are equal.*

2. *All right angles are equal.*

For a right angle is half a straight angle. (See Def. 24.)

3. *The complements of equal angles are equal.*

1. Define *right angle*, *adjacent angles*.

2. Let the straight line AB stand upon the straight line CD so as to make the adjacent angles ABC and ABD equal. Draw BE dividing the angle ABC into any two parts. Prove

(i) That the angle EBD is obtuse.

(ii) That the sum of the angles CBE , EBA and ABD is equal to a straight angle.

III.—Theorem 2.—(Euc. I. 13.)

If a straight line stands upon another straight line, then the adjacent angles are together equal to two right angles.

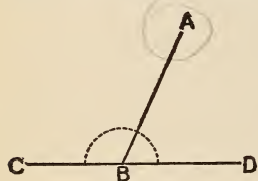


FIG. 63.

Given. That the straight line AB stands upon the straight line CD .

To be proved. That the adjacent angles ABC and ABD are together equal to two right angles.

Proof. Since CBD is a straight line,
Given.

the angle CBD is a straight angle ;

Def.22.

therefore the angle CBD is equal to two right angles. *Def.24.*

But the angle CBD is equal to the angles ABC and ABD taken together ;

Ax.9.

therefore the angles ABC and ABD are together equal to two right angles.

Ax.1.

Q.E.D.

IV.—Theorem 3.—(Euc. I. 14.)

If at a point in a straight line two other straight lines on opposite sides of it make the adjacent angles together equal to two right angles, then these two straight lines are in one and the same straight line.

Given. That at the point B in the straight line AB the straight lines BC and BD, on opposite sides of AB, make the adjacent angles ABC and ABD together equal to two right angles.

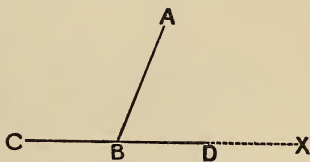


FIG. 64.

To be proved. That BC and BD are in the same straight line.

Construction.*—Let CB be produced to X.

Proof. Because CBX is a straight line, *Const.*
the adjacent angles ABC and ABX are together equal to two right angles. *A.2—I.13.*

But the angles ABC and ABD are together equal to two right angles. *Given.*

Therefore the angles ABC and ABX are together equal to the angles ABC and ABD. *Cor.2, A.1 and Ax.1.*

Take away from each of these equals the common angle ABC; then the remaining angle ABX is equal to the remaining angle ABD. *Ax.3.*

Therefore BD falls along BX;
that is, BD is in the same straight line with BC. **Q.E.D.**

* The construction of a theorem describes any lines, etc., drawn to enable us to prove the truth of the theorem.

V.—Theorem 4.—(Euc. I. 15).

If two straight lines cut one another, then the vertically opposite angles are equal.

Given. That the straight lines AB and CD cut one another at the point E .

To be proved. That the vertically opposite angles AEC and BED are equal; also, that the vertically opposite angles AED and BEC are equal.

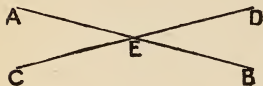


FIG. 65.

Proof. Because CE stands upon AB , therefore the adjacent angles AEC and BEC are together equal to two right angles. *A.2—I.13.*

Again, because BE stands upon CD , therefore the adjacent angles BED and BEC are together equal to two right angles. *A.2—I.13.*

Therefore the angles AEC and BEC are together equal to the angles BED and BEC . *Ax.1.*

Take away from these equals the common angle BEC , then the remaining angle AEC is equal to the remaining angle BED . *Ax.3.*

Similarly it may be shown that the angle AED is equal to the angle BEC . *Q.E.D.*

COROLLARY :

If any number of straight lines in a plane meet at a point, the sum of all the angles formed by these lines, each with the next in order, is equal to four right angles.

1. ABC is an obtuse angle. AB and CB are produced to D and E . Prove that the angle DBE is obtuse.

2. Let X be any point in the straight line PQ . On opposite sides of PQ let the straight lines XY and XZ be drawn so as to make the angles PXY and QXZ equal. Show that XY and XZ are in the same straight line.

VI.—Indirect Demonstration.

A theorem may be true and yet it may be difficult or even impossible to prove directly that it is so. On the other hand it may be quite easy to show that any statement which contradicts the theorem is false. Here are two statements which contradict each other :

An angle can not have more than one bisector.

An angle can have more than one bisector.

That such a thing as a bisector of an angle can exist we have already admitted, otherwise we should not have defined it. (See Def. 21.) The question to be settled is whether there can be more than one or not.

We shall now suppose the second statement to be true, and afterwards we shall consider whether the conclusion to which this supposition leads us, is reasonable or absurd.

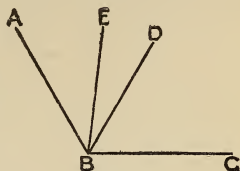


FIG. 66.

Suppose $\angle ABC$ to be an angle of which BD and BE are two bisectors. Let BE fall between BA and BD .

Since BD is a bisector of the angle ABC , the angle ABD is half of the angle ABC . Def. 21.

Again, since BE is a bisector of the angle ABC , the angle ABE is half of the angle ABC . Def. 21.

Therefore the angle ABD is equal to the angle ABE . Ax. 7.

But the angle ABE is only a part of the angle ABD , therefore these angles are unequal. Ax. 9.

We have thus proved that the angles ABD and ABE are both equal and unequal, which is absurd.

By supposing that an angle can have more than one bisector, we have been compelled by sound reasoning to make a most unreasonable statement, viz., that two things can be both equal and unequal at the same time. Now we know that an absurd conclusion cannot be a necessary consequence of a supposition that is true, for then one truth would contradict another, which is impossible. We therefore conclude that we made a false supposition. Having thus shown that the second statement is false, we have proved conclusively that the first is true.

When the truth of a theorem is proved indirectly by showing that any supposition which contradicts it leads to an absurd conclusion, the form of demonstration is called *reductio ad absurdum*.

Let **AB** and **CD** be two given straight lines. Consider the following statements :

(a) *AB is greater than CD.*

(b) *AB is equal to CD.*

(c) *AB is less than CD.*

Observe that one of these statements must be true ; also, that only one of them can be true.

Suppose certain facts to be known or given whereby we can prove that (a) is not true, then it follows that either (b) or (c) is true. Again, if we can prove that (b) is not true, we can say with certainty that the remaining statement (c) is true. This form of demonstration is called *proof by exhaustion*.

The validity of an indirect demonstration depends upon the fulfilment of two conditions :

(i) Of two or more assertions *one is necessarily true*.

(ii) *With one exception*, all of these assertions are proved to be false.

VII.—Theorem 5.

An angle can have only one bisector.

The proof of this theorem is given on the preceding page. Write an explanation of the argument.

COROLLARY :

At a given point in a given straight line only one perpendicular to it can be drawn in a plane.

VIII.—Theorem 6.

A straight line can have only one point of bisection.

Outline.—Let **AB** be a given straight line which is bisected at the point **P**. Suppose that **AB** has another point of bisection **Q**. This supposition is false ; hence, the conclusion.

Write the argument in full.

CHAPTER VI.

CIRCLES.

Exercise XXVI.*

1. Take a point P on the surface of a sheet of paper. Find five points which shall be equidistant from P.

2. Trace a line, every point in which shall be the same distance from the point P.

3. Define *circle*, *circumference*, *radius*.

4. Draw a straight line 1 in. long. Without further measurement find six points each 4 in. from one of its extremities.

5. Draw a straight line which shall be twice the length of the straight line AB. Three times the length of AB. Four times the length of AB. Give reasons in each case.

6. Produce the straight line PQ so that the part produced shall be four times PQ.

7. Find a point equidistant from the points A and B. Find another.

8. Draw a straight line AB 1 in. long. Find two points each of which shall be 2 in. from both A and B.

9. Describe a circle. Draw, using protractor, two radii forming an angle of 60° . What part of the whole circle is included by these radii and the arc which they intercept? Why?

10. Divide, using protractor, a semicircle into six equal sectors. What is the measure of each of the angles formed by the radii? What part of the semicircumference is each of the arcs?

11. Define *sector*, *quadrant*. Illustrate by diagrams.

* Henceforward all constructions will be effected, as far as possible, by means of the ungraduated ruler, compasses, and pencil. See foot note, page 26.

Exercise XXVII.

1. Describe a circle. Draw any radius. Through the point where the radius meets the circumference, draw three straight lines as follows :

- (i) One which divides the circle into two equal parts.
- (ii) One which divides the circle into two unequal parts.
- (iii) One which does not divide the circle into parts.

2. Define *segment*, *semicircle*. Illustrate.

3. Describe a circle. From any point without it draw straight lines as follows :

- (i) One which cuts the circumference at two points as far apart as possible.
- (ii) One which meets the circumference at only one point.

4. Define *secant*, *chord* and *diameter*. How are they related to one another? Define *tangent*.

5. Describe a circle whose diameter is $2\frac{1}{2}$ in. long. Place in it a chord 1 inch long.

6. Draw two circles whose radii are equal. Show that the circles are equal.

7. Describe a circle. Draw two sectors having equal angles at the centre. Show that if one be applied to the other their arcs must coincide.

8. Show by superposition that if two arcs of a circle be equal, the angles formed by the radii intercepting them must be equal.

[Two arcs are said to be equal when they can be made to coincide with each other.]

9. What part of a whole circle does each of the following sectors constitute :

- (i) A sector whose angle is 90° ?
- (ii) A sector whose angle is 60° ?
- (iii) A sector whose angle is 30° ?

Draw these, using a protractor.

10. Fifteen equal angles are formed by straight lines drawn from the centre of a circle to meet the circumference. Find the measure of these angles :

- (i) In degrees.
- (ii) In terms of a right angle.

11. Draw any ornamental design composed of concentric circles and arcs passing through their common centre.

12. Draw any design composed of tangent circles.

EXPLANATIONS.

I.—Circle.—Definitions.

Let AB in Fig. 67 be a straight line of given length which rotates in a plane about the point A, as indicated by the arrow.

The following facts will be observed :

(i) That the path of the point B is a line, every point in which is at a distance equal to the length of AB from the fixed point A.

(ii) That the point B returns to its initial position after making a complete revolution about the fixed point. If the motion be continued further, the point B will retrace its former path.

(iii) That the straight line AB in making a complete rotation about the fixed point (being moved in a plane) describes a plane surface, which is limited by the line traced out by the point B.

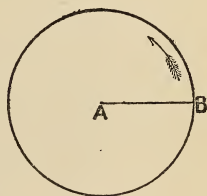


FIG. 67.

The plane surface described by the straight line AB, in making a complete rotation about the fixed point is called a **circle**. The surface described by AB in making less than one rotation is a **sector**. The line limiting the circle is its **circumference**, and the fixed point its **centre**. The straight line AB, in any of its positions, is a **radius** of the circle.

Any portion of a circumference is termed an **arc**.

The characteristic by which the circle is distinguished from every other plane surface enclosed by a curved line, is that *all points in the boundary of the circle are equidistant from a fixed point within it*.

The distance of any point on the circumference of a circle from the fixed point, or centre, is called the *radius-distance*.

II.—Measurement of Angles.

Draw a circle. Set off any number of equal arcs. Join their extremities to the centre of the circle. Compare the angles standing on these arcs.

If the circumference of a circle is divided into any number of equal arcs, and if radii are drawn to their extremities, the angles formed at the centre of the circle by these radii, each with the next in order, are equal to one another.

For purposes of practical measurement it has been found convenient to divide the whole circumference of a circle into equal parts, and to regard an arc thus obtained as the measure of the angle formed by the radii intercepting it. An arc which is $\frac{1}{360}$ of the circumference is called a **DEGREE** ($^{\circ}$).

III.—Drawing Circles.

In drawing circles a pair of compasses, as shown in Fig. 68, may be used. The compasses, being set to the required radius-distance, will be placed on the paper so that the fine point at the extremity of one leg coincides with the given centre of the circle to be drawn. The marking point will then be moved about the centre until the circumference is described.



FIG. 68.

IV.—Meaning of Problem.

A statement of any geometrical construction, such as drawing a line, erecting a perpendicular, bisecting an angle, etc., to be effected by means of certain instruments, is called a **problem**.

A problem consists of two parts :

- (i) The *data*, or *things given*.
- (ii) The *quæsitæ*, or *things required*.

NOTE.—It is quite allowable to use a scale, protractor, set-square, or other suitable instrument in making any construction corresponding to the *data* of a geometrical problem ; but no instrument other than the straight-edge, compasses and pencil (or drawing pen) should be used in effecting the construction demanded by the *quæsitæ*.

The following are the principal steps in solving a geometrical problem :

The first step is to represent by means of a diagram the data of the problem.

The second is to find out, if possible, how to proceed in making the construction required. We suppose that the construction to be made already exists, and that it is before us for examination. We then examine it carefully, noting any of its properties we may discover. This “analysis of the problem” paves the way for, and should therefore precede the work of constructing the figure required by the *quæsitæ* of the problem.

The third is to effect the required construction by means of instruments. Each exercise should be repeated until it can be done neatly and accurately. Drawings should frequently be tested by means of “check” constructions.

The fourth step is to prove that the construction effected conforms to the requirements of the problem.*

V.—Fundamental Problems.

1. *To find any number of points which are at a given distance from a given point.*

ANALYSIS.—Take any point P in a plane. Suppose the points A, B, C, D, E, etc., in the plane to be at the distance d from P. Then the

* Theoretical proofs will be required when the theorems upon which they are based are demonstrated.

straight lines PA , PB , etc., are equal. Why? Where must the points A , B , C , etc., lie?

Let P be the given point, and let the length of the straight line AB be the given distance: it is required to find any number of points at the distance AB from P .

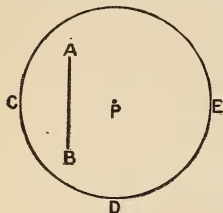


FIG. 69.

With centre P and radius equal to AB , describe the circle CDE . Then all points in the circumference CDE are at the given distance AB from P .

2. *To find any number of points which are equidistant from two given points.*

ANALYSIS.—Take any two points P and Q . Suppose the point A to be equidistant from P and Q , then PA is equal to QA . If A is not in PQ then it is obvious that some other point on the opposite side of PQ must be at the same distance as A from P and Q . Let C be that point. Then PC is equal to PA , and QC is equal to QA . What construction does this suggest?

Let P and Q be the given points: it is required to find any number of points which are equidistant from P and Q .

With centre P and radius greater than one-half the length of the straight line PQ , describe the arc ABC . With centre Q and radius the same as before, describe the arc ADC cutting the arc ABC at A and C . Join AC . Then all points in AC are equidistant from P and Q .

Draw arcs by which the accuracy of the construction may be tested. Explain this test.

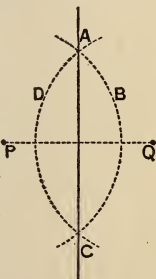


FIG. 70.

3. *To bisect a given finite straight line.*

Let AB be the given finite straight line: it is required to bisect it.

Let C be a point in AB such that AC is greater than one-half of AB . With centre A and radius AC describe arcs on opposite sides of AB . With centre B and radius the same as before, describe arcs cutting the other arcs at the points D and E . Join DE . Then AB is bisected at O , the point at which AB and DE intersect.

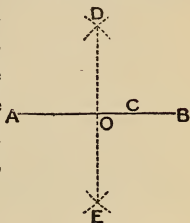


Fig. 71.

4. *To draw a straight line at right angles to a given straight line from a given point in it.*

Let AB be the given straight line, and let P be the given point in it: it is required to draw from P a straight line at right angles to AB .

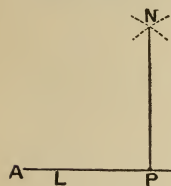


Fig. 72.

Take any point L in PA . From PB , or PB produced, cut off PM equal to PL . With L and M as centres, and radius greater than one-half of LM , describe arcs intersecting at N . Join PN . Then PN is at right angles to AB .

5. *To draw a straight line at right angles to a given straight line from one of its extremities.*

HINT.—Take a point in the arc of a semicircle. Join it to the extremities of the diameter. Measure with protractor the angle formed by the straightlines thus drawn. Take another point, as before, and so on. What construction does this suggest?

Let AB be the given straight line: it is required to draw from the point B a straight line at right angles to AB .

Take any point O outside of AB . With centre O and radius OB describe the arc CBD , cutting AB at C . Join CO and produce it to meet the arc at E . Join BE . Then BE is at right angles to AB .

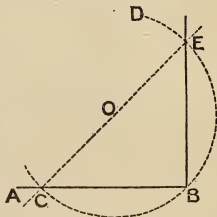


Fig. 73.

6. *To draw a straight line perpendicular to a given straight line from a given point without it.*

Let AB be the given straight line, and let P be the given point without it: it is required to draw from P a straight line perpendicular to AB .

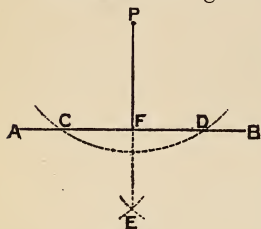


FIG. 74.

From the centre P at any sufficient distance describe an arc cutting AB at C and D . With C and D as centres and radius greater than one-half of CD describe arcs intersecting at E . Draw PE cutting AB at F . Then PF is perpendicular to AB .

NOTE.—The **distance** of a point from a straight line is the length of the perpendicular drawn from the point to the line. Thus in Fig. 74 the length of PF is the distance from P to AB .

7. *To bisect a given angle.*

Let BAC be the given angle: it is required to bisect it.

In AB take any point D . With centre A and radius AD describe an arc cutting AB and AC at D and E . With D and E as centres and radius greater than one-half the distance between D and E describe arcs intersecting at F . Join AF . Then AF bisects the angle BAC .

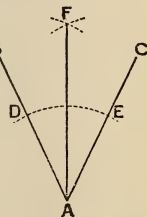


FIG. 75.

8. *At a given point in a given straight line to make an angle equal to a given angle.*

HINT.—As the vertex and one arm of the required angle are given, the problem consists in finding a second point (the vertex being one) through which the other arm must pass.

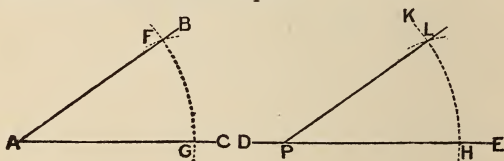


FIG. 76.

Let DE be the given straight line, P the given point in it, and BAC the given angle. It is required to draw from P a straight line making with DE an angle equal to BAC .

In AB take any point F . With centre A and radius AF describe an arc cutting AC at G . With centre P and radius equal to AF describe an arc HK cutting DE at H . With centre H and radius equal to GF^* describe an arc cutting the arc HK at L . Join PL . Then the angle LPH is equal to the given angle BAC .

9. To find the centre of a given circle.

ANALYSIS.—With centre P and any radius describe the circle LMN . Draw any diameter XY . Draw a chord perpendicular to XY . If the position of XY were unknown, show that it could be found by means of the chord.

Let ABC be the given circle: it is required to find the centre of the circle ABC .

Draw any chord AB . Draw DE bisecting AB at right angles, the points D and E being on the circumference of the given circle.

Bisect DE at O . Then O is the point required.

10. Through a given point on the circumference of a circle to draw a tangent to the circle.

HINT.—Through the given point draw a straight line perpendicular to the radius at that point.

11. Through a given point without a circle to draw a tangent to the circle.

HINT.—The construction is based on the fact that the angle in a semi-circle is a right angle. Join the centre of the circle to the given point, etc. (See Problems 5 and 10.)

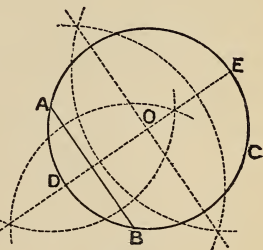


FIG. 77.

VI.—Meaning of Locus.

Describe a circle whose centre is P and whose radius-distance is 1 inch. Observe:

* GF denotes the straight line joining the points G and F . See Def. 18, page 41.

(a) Every point in the circumference satisfies one condition ; it is at the fixed distance of one inch from the given point P.

(b) Every point in the plane not in the circumference fails to satisfy this condition. For if it is either without or within the circumference, evidently its distance from P is greater than, or less than 1 inch.

Now if we know that any point X in a plane is at the distance d from the fixed point O in the plane we can find the line in which X lies—the circumference of the circle whose centre is O, and whose radius-distance is d . This line is said to be the *locus* of the point X.

It will be observed that the given condition does not enable us to locate the point X. All we can say about it is that it lies somewhere in a circumference whose position is known.

The *locus* (plural, *loci*) of a point satisfying a given condition is the line, or group of lines, containing all the points which satisfy the given condition, and no other points.

VII.—Fundamental Loci.

1. To find the locus of a point which is at a given distance from a given point.

(For the method of finding this locus, see Problem 1, page 77.)

2. To find the locus of a point which is equidistant from two given points.

(See Problem 2, page 78.)

VIII.—Intersection of Loci.

Let X be a point which satisfies the following conditions :

(i) It lies in the given straight line AB.

(ii) It is at the distance d from the fixed point P.

Referring to Fig. 78, we know that according to the first condition X must lie somewhere in AB. But this condition alone does not enable us to locate the point X, for an unlimited number of points lie in the locus AB.

Again, according to the second condition X must lie in the circumference of the circle CDE. But an unlimited number of points lie in this locus also ; therefore we cannot locate X by considering the second condition by itself.

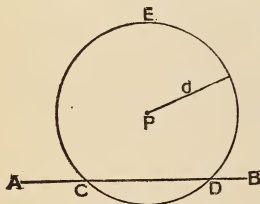


FIG. 78.

Let us next consider where X must lie in order to satisfy both con-

ditions. Evidently its position must be that of C or D, for these are the only two points which are in the straight line AB, and also in the circumference CDE. Hence there are two positions of the point X and only two, which satisfy both given conditions.

When two loci of a point are known the position of the point is determined by the intersection of these loci.

Let P and Q be two fixed points at a distance of $2\frac{1}{2}$ in. apart. By the *Method of Loci* determine the position of a point $1\frac{1}{2}$ in. from P and 2 in. from Q. How many positions can this point take? Would a solution be possible if P and Q were 5 in. apart? Illustrate.

Find a point which is at the distance c from the given point P, and also equidistant from the given points Q and R.

Find a point which is equidistant from three given points.

DEFINITIONS.

31. A plane figure is a part of a plane enclosed by a line, or lines.

The boundaries of a plane figure may be straight lines, curved lines, or both straight and curved lines.

32. The quantity of surface enclosed by the boundary, or boundaries, of a plane figure is called its area.

Plane figures of the same size, but not of the same shape, are said to be *equivalent*, or *equal in area*.

Plane figures which can be placed so as to coincide with one another, are said to be *congruent*. Such figures agree in both size and shape.

Congruent figures are, by the axiom of superposition, **identically equal**, that is to say, they are *equal in all respects*.

33. A circle is a plane figure contained by one line, which is called the *circumference*, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the *centre* of the circle.

34. An arc is any part of a circumference.

35. A radius (plural, *radii*) is a straight line drawn from the centre to the circumference.

36. A **sector** is a figure contained by two radii and the arc which they intercept.

A sector whose limiting radii form a right angle is called a **quadrant**.
[An angle formed by any two radii is termed *an angle at the centre*.]

37. A **secant** is a straight line of unlimited length, which cuts the circumference of a circle at two points.

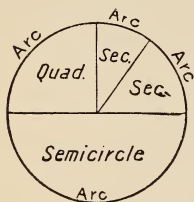


FIG. 79.

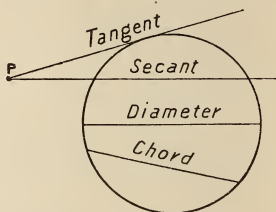


FIG. 80.

38. A **chord** is that part of a secant which is terminated both ways by the circumference.

39. A chord which passes through the centre of a circle is called a **diameter**.

40. A **tangent** is a straight line of unlimited length which meets a circle at only one point.

If the line marked "secant" in Fig. 80 be rotated in the plane of the circle about the point P towards the line marked "tangent" the two points at which the moving line cuts the circumference will approach each other. In one of its positions—the limiting position of the secant—these points will coincide. When in this position, the secant is tangent to the circle.

The point at which a tangent meets a circle is called the *point of tangency*, or *point of contact*.

41. A chord divides a circle into two parts, each of which is called a **segment**.

When the segments are equal—that is, when the chord is a diameter, each segment is a *semicircle*.

42. Concentric circles are such as have a common centre.

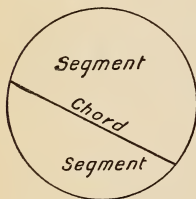


FIG. 81.

43. Circles whose circumferences meet at only one point are said to *touch* one another. Such circles are usually called **tangent circles**.

AXIOM.

B. A line drawn between two points, one being within, and the other without a closed figure, cuts the boundary of the figure.

[This truth is assumed by Euclid, though not stated as an axiom.]

From Axiom B deduce the following :

(i) A straight line of unlimited length drawn through a point within a closed figure cuts the boundary of the figure at two points at least.

(ii) If there are any two points in the boundary of a closed figure, such that one lies within and the other without another closed figure, the boundaries of these figures intersect at two points at least.

POSTULATE.

III. A circle may be described from any centre and at any distance from that centre.

This postulate states the use to be made of compasses in drawing geometrical figures. It does not, strictly speaking, allow us to transfer distances, except in so far as such may be effected in a plane by moving one extremity of the compasses, the other being fixed. Theoretically, there can be no good reason for this restriction, as it has been already assumed that magnitudes may be moved in space without alteration of shape or size. Practically, compasses are used freely for transferring distances.

THEOREMS.

SECTION B—CIRCLES.

I.—Theorem 1.

The distance of a point from the centre of a circle, is greater than, equal to, or less than the radius, according as the point lies without, on, or within the circumference.

Given. That ABC is a circle and P a point without its circumference.

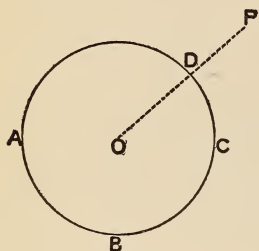


FIG. 82.

To be proved. That the distance OP is greater than the radius, O being the centre of the circle ABC .

Construction. Draw the straight line OP .

Proof. Because O is the centre of the circle ABC , and P lies without its circumference, *Given.*

OP must cut the circumference. *Ax.B.*

Let OP cut it at the point D .

Since OP is a straight line of which OD is a part, OP is greater than OD . *Ax.9.*

Therefore the distance OP is greater than the radius OD .

(Prove the other cases.)

From Theorem 1, deduce the following :

The locus of a point at a given distance from a fixed point, is the circumference of the circle whose radius-distance is the given distance, and whose centre is the fixed point.

II.—Converse Theorems.

When two theorems are such that the *hypothesis* of each is the *conclusion* of the other, the theorems are said to be **converse** of each other.

For example, each of the theorems, “If the distance of a point from the centre of a circle is greater than the radius, then the point lies without the

circumference," and "If a point lies without the circumference of a circle, then its distance from the centre is greater than the radius," is the converse of the other.

Stated generally, the theorems, "If A is B , then C is D ," and "If C is D , then A is B ," are converse theorems.

Resolve Theorem 1 into three separate theorems, and write the converse of each.

Although it often happens that a theorem and its converse are both true, it must be kept in mind that this is not necessarily the case. To prove the truth of a theorem is, therefore, not sufficient to establish the truth of its converse.

III.—Theorem 2.

If the radii of two circles are equal, the circles are identically equal.

Outline.—Let ABC and LMN be two circles of equal radii, whose centres are P and Q respectively. Apply the circle ABC to the circle LMN so that P may fall on Q . Take any point X outside of both circles. Join QX . QX will cut both circumferences at the same point. The circumferences coincide at all points ; hence, the conclusion.

Write the argument in full.

COROLLARIES :

1. *The circumferences of two concentric circles of unequal radii can neither coincide with, nor cut one another.*

2. *Hence if two circles are equal, their radii are equal.*

NOTE.—In proving the equality of two figures A and B by the *Method of Superposition*, the argument is as follows :

(a) *Magnitudes that can be made to coincide with one another, are equal.*

(b) *The figures A and B can be made to coincide with each other.*

(c) *Therefore A and B are equal.*

Now since (a) is always true, and it includes all magnitudes that may be compared by superposition, evidently the validity of the argument must depend upon proving conclusively that (b) is true. It must be shown that, according to the given conditions, the boundaries of A and B will coincide at all points. This being done, it immediately follows that A and B are equal in all respects.

IV.—Theorem 3.—(Euc. III. 26.)

In the same circle, or in equal circles, the arcs which subtend equal angles at the centre, are equal.

Given. That the circles ABC and LMN are equal; also, that the angles BPC and MQN at their centres are equal.

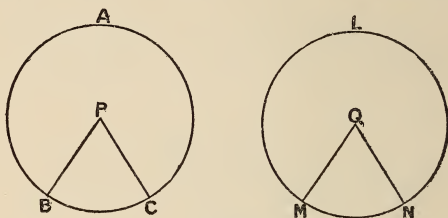


FIG. 83.

To be proved. That the arc BC is equal to the arc MN .

Proof. Apply the circle ABC to the circle LMN , so that the centre P may be on the centre Q , and the radius PB may fall along the radius QM .

Because the angle BPC is equal to the angle MQN , *Given*, therefore the radius PC must fall along the radius QN .

Again, because the circles ABC and LMN are equal, *Given*, therefore their radii are equal. *B.2., Cor.2.*

Hence B must fall on M , and C on N ; also, every point in the arc BC must fall on the arc MN ; that is, the arc BC must coincide with the arc MN .

Therefore, the arc BC is equal to the arc MN . *Ax.8.*
Q.E.D.

COROLLARIES :

1. *In the same circle, or in equal circles, sectors which have equal angles are equal.*

2. *In the same circle, or in equal circles, sectors which are equal, have equal arcs.*

3. *Every diameter of a circle divides it into segments which are identically equal.*

V.—Theorem 4.—(Euc. III. 27.)

In the same circle, or in equal circles, the angles at the centre which stand on equal arcs, are equal.

- (i) Write an outline of the argument.
- (ii) Write the argument in full.
- (iii) State the converse of this theorem.

COROLLARIES :

1. *In the same circle, or in equal circles, sectors which have equal arcs are equal.*

2. *In the same circle, or in equal circles, sectors which are equal have equal angles.*

1. Show that a circle cannot have more than one centre. (Indirect proof.)

2. Prove the converse of Theorem I. (Indirect proof.)

3. Show that if the circumferences of two circles cut one another the circles cannot be concentric.

4. *The quadrants of a circle are identically equal.*

5. *In the same circle, or in equal circles, if two arcs are unequal, that angle at the centre is the greater which stands on the greater arc.*

6. *In the same circle, or in equal circles, if two angles at the centre are unequal, the greater angle stands on the greater arc.*

 PROBLEMS.

1. Draw a straight line **AB**. Produce it to **C**, making **BC** equal to **AB**.

2. Find a straight line which shall be three times as great as the given straight line **PQ**.

3. Find a straight line equal to the sum of the given straight lines **AB** and **AC**. Equal to their difference. Equal to twice their difference.

4. Describe a circle whose radius shall be equal to twice the difference of the given straight lines **PQ** and **RS**.

CHAPTER VII.

TRIANGLES.

Exercise XXVIII.

1. Point out objects bounded by plane surfaces. Plane surfaces bounded by three, four, five, and six straight lines, respectively.

2. Define *rectilineal figure*. Draw six rectilineal figures.

3. Take any three points on a sheet of paper not in the same straight line. How many straight lines can be drawn through these points two and two? Draw the lines. Describe the enclosed figure.

4. Define *triangle*.

5. Draw a triangle having two sides equal.

6. Draw a triangle having its three sides unequal.

7. Draw a triangle, making each of the angles at the base 60° . (Use protractor.)

8. Define *equilateral triangle*, *isosceles triangle*, and *scalene triangle*.

9. Draw a triangle whose sides are respectively 2 in., $1\frac{1}{2}$ in., and 1 in. long.

10. Is it possible to draw a triangle whose sides are respectively 2 in., 1 in., and $\frac{1}{2}$ in. long? Illustrate.

Exercise XXIX.

1. Draw a triangle having its base 3 in. long, and each of the angles at the base an angle of 60° . Find by measurement the sum of the angles of the triangle. (Use protractor.)

2. Draw a triangle ABC, making AB 3 in. long, and the angle ACB a right angle. Find by measurement the sum of the angles of the triangle.

3. Draw a triangle whose sides are $1\frac{1}{2}$ in., 2 in., and $2\frac{1}{2}$ in. long. Measure each of its angles. Find the sum of all its angles.

Produce the sides in the same order, and measure each of the three exterior angles thus formed. Find the sum of the exterior angles.

4. Draw three triangles differing as much as possible in shape. Cut out. Divide each into three pieces, and arrange the pieces in each case as shown in Figs. 84 and 85.

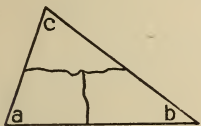


FIG. 84.

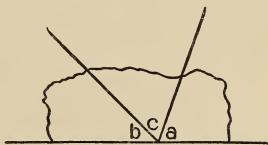


FIG. 85.

What conclusion do you come to as to the sum of the three angles of a triangle?

5. State all the facts learned regarding interior and exterior angles of triangles.

6. How many interior angles of a triangle may be acute? How many obtuse? How many may be right angles?

7. Define *acute-angled triangle*, *obtuse-angled triangle*, and *right-angled triangle*. Draw one of each.

8. Define *hypotenuse*. Illustrate.

9. Describe a circle. Draw any diameter. Join the extremities of the diameter to any point in either arc. Examine the triangle thus formed.

10. Draw two other triangles as in No. 9. Measure the angles of each. Classify these triangles.

11. Draw a right-angled triangle having the hypotenuse 3 in. long and one of its acute angles $\frac{1}{3}$ of a right angle.

Exercise XXX.

1. Draw a triangle ABC, making each of the angles ABC and BAC $\frac{2}{3}$ of a right angle. Without measurement, determine the size of the angle ACB.

2. The vertical angle of an isosceles triangle ABC is $\frac{1}{3}$ of a right angle, and the length of the base AB is 1 in. Draw the triangle.

3. Calculate the value in degrees of the angles of the following isosceles triangles :

(i) The vertical angle is $\frac{2}{3}$ of a right angle.

(ii) One of the angles at the base is $\frac{5}{6}$ of a right angle.

4. Find the value in degrees of each of the exterior angles of the triangles in No. 3.

5. Draw a triangle having two of its sides $2\frac{1}{2}$ and 2 in. long respectively, and the angle included by these sides 60° .

(i) State the data.

(ii) State what is required to be done.

(iii) Draw a second triangle in accordance with the same data.

(iv) Examine the two figures, and say whether they are of the same shape and size or not. Test by superposition.

6. Two straight lines PQ and PR meet at the point P, forming an angle of 60° . PQ and PR are each 12 ft long. Draw to a scale of $\frac{1}{4}$ in. per ft., and find by measurement of diagram the distance from Q to R.

7. Measure the length and breadth of the room. Draw on blackboard to a scale of 1 in. per ft. Find the distance between opposite corners of the room by measurement of diagram. Test.

8. PQR is a triangular field. The distance from P to Q is 30 rods, and from P to R 25 rods. The angle QPR is 60° . Owing to an obstruction, as in Fig. 86, it is impossible to measure QR directly. Devise a means of finding the length of QR.

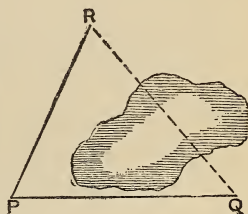


FIG. 86.

9. AB is a pole set vertically into the ground, C is a point on the ground at a distance of 20 ft. from B at the foot of the pole. The angle ACB is 60° . Lay out a triangle by means of which the height of the pole may be determined.

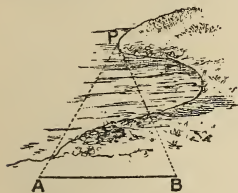


FIG. 87.

10. A and B are two accessible points, and P is a third point which is inaccessible, as in Fig. 87. Show what measurements of the triangle ABP must be made in order that another triangle may be laid out on level ground so as to be equal in all respects to ABP.

EXPLANATIONS.

I.—Rectilineal Figure.

Let A, B, C.... be any number of points in plane, as in Fig. 88.

Join AB, BC, etc., so as to enclose a portion of the plane.

On examining the diagram we find that the whole plane is divided into two parts:

(i) The *limited* part, which is bounded by the straight lines AB, BC, CD, DE, EF, and FA.

(ii) The *unlimited* part, which lies without the boundary of the other.

That part of the plane, which is bounded by the straight line AB, BC, etc., is called a **plane rectilineal figure**.

The quantity of surface which it contains is called its **area**.

The straight lines forming its boundary are called its **sides**.

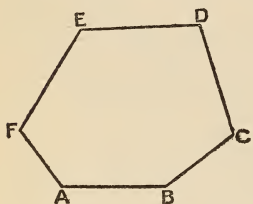


FIG. 88.

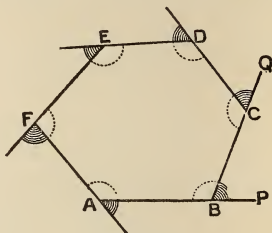


FIG. 89.

The angles ABC, BCD, etc., formed by the sides of a figure, as indicated by the dotted lines in Fig. 89, are called the **interior angles**, or more briefly, the **angles** of the figure.

The angles PBC, QCD, etc. (shaded in diagram) formed by producing the sides in the same order, are called the **exterior angles** of the figure.

Each of the points A, B, C is called a **vertex** (plural, *vertices*) of the figure.

II.—Triangle.

Let A, B and C be any three points not in the same straight line in a plane. Join AB, BC and CA. The three straight lines thus drawn enclose a part of the plane, as in Fig. 90.

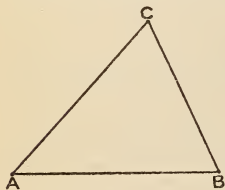


FIG. 90.

The plane surface bounded by these three straight lines is called a **triangle**.

What has been said above regarding the boundaries, angles, and vertices of plane rectilineal figures is applicable to triangles.

The component parts, or elements, of a triangle are :

- (i) Its *three sides*.
- (ii) Its *three angles*.
- (iii) Its *area*.

Triangles are classified (*a*) according to sides, (*b*) according to angles. (See definitions.)

III.—Sides of a Triangle.

Any two sides of a triangle are together greater than the third side.

In the triangle ABC, as in Fig. 90, the straight line AB is the shortest that can be drawn from A to B.* Therefore, AB must be less than the broken line ACB; that is, AC and CB taken together are greater than AB.

Show that the sum of AB and BC is greater than AC; also, that the sum of AC and AB is greater than BC.

IV.—Angles of a Triangle.

The sum of the exterior angles of a triangle (formed by producing its sides in the same order) is equal to two straight angles, or four right angles.

Let ABC be any triangle. Let the sides AB, BC and CA be produced to D, E, and F, as in Fig. 91.

The angles CBD, ACE and BAF thus formed are called the exterior angles of the triangle ABC.

Suppose an unlimited straight line PQ† placed along AD to turn (as indicated by the arrows) about the point B until it lies along BE; then about C until it lies along CF; then about A until it again lies along AD—its initial position.

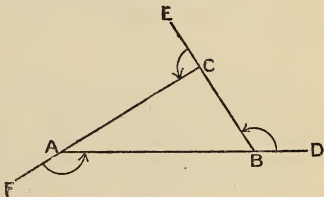


FIG. 91.

Since PQ has turned successively through the exterior angles CBD, ACE, and BAF, their sum is equal to the whole angle through which PQ has turned; that is, their sum is equal to two straight angles or four right angles.

* The truth illustrated on page 34, under "Distance," is here assumed (provisionally) as an axiom.

† The movable straight line PQ may be represented by a straight-edge or fine cord.

Also, the sum of the angles (interior) of a triangle is equal to one straight angle, or two right angles.

(i) Deduce this truth from the foregoing.

(ii) Prove it by turning a straight line through the angles of a triangle in succession.

V.—Equality of Triangles.

Let A, B and C be any three fixed points not in the same straight line in a plane. Through these points, two and two, draw straight lines, as in Fig. 92.

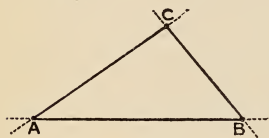


FIG. 92.

If we draw another set of straight lines through the points A, B and C, two and two, these lines will fall upon those already drawn; consequently, no new triangle will be formed.

We thus see that *three fixed points not in the same straight line in a plane, determine the shape, size, and position of a triangle.*

It follows from the foregoing that if we can locate the vertices of any required triangle we can construct the triangle; also, if we can make the vertices of one triangle coincide with those of another, the triangles must coincide, for the two triangles will then constitute one and the same triangle.

To make a triangle that will coincide with any given triangle ABC.

(i) Measure AB (as in Fig. 92) and make PQ equal to it.

(ii) Measure the angle BAC and make the angle QPR equal to it.

(iii) Measure AC and from PR (produced if necessary) cut off PS equal to AC. Join QS.

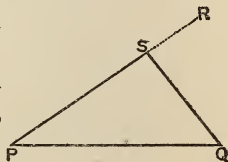


FIG. 93.

We have thus constructed a triangle PQS that will coincide with the given triangle ABC. These triangles are, therefore, identically equal.

Review the foregoing and state definitely :

(a) In what respects the triangle PQS is made equal to the triangle ABC.

(b) In what respects the triangle PQS will coincide with the triangle ABC, giving reasons.

(c) In what respects the triangles are equal.

Hence, if two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, then the triangles are equal in all respects.

NOTE.—When two triangles are equal in all respects, the equal sides are opposite to the equal angles; and conversely, the equal angles are opposite to the equal sides.

VI.—Fundamental Problems.

11. To describe an equilateral triangle on a given finite straight line.

Let AB be the given finite straight line.

It is required to describe an equilateral triangle on AB.

With centre A and radius AB describe the arc PQ. With centre B and radius BA describe the arc RS. Let C be the point at which the arcs intersect.

Join AC and BC. Then ABC is the triangle required.

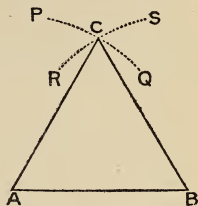


FIG. 94.

12. To construct a triangle having its sides equal to three given straight lines, respectively.

ANALYSIS.—Suppose LMN to be the triangle required, LM, LN and MN corresponding to the given straight lines AB, CD and EF respectively. The vertical points L and M are at what distance apart? At what distance from L is the vertex N? Find any number of points at that distance from L, etc.

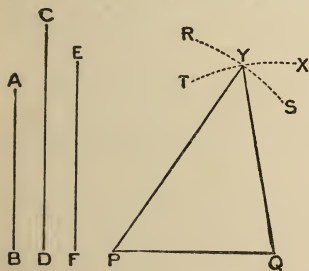


FIG. 95.

Draw PQ equal to AB.

Let AB, CD and EF be the three given straight lines.

It is required to construct a triangle having its sides equal to AB, CD, and EF respectively.

With centre P and radius equal to CD, describe the arc RS. With centre Q and radius equal to EF, describe the arc TX. Let Y be the point at which the arcs intersect. Join PY and QY. Then PQY is the triangle required.

13. *To construct a triangle having two of its sides equal to two given straight lines respectively, and the angle included by these sides equal to a given angle.*

14. *To construct a triangle having its base equal to a given straight line, and the angles at the base equal to two given angles, respectively.*

15. *To construct a triangle having two sides equal to two given straight lines, respectively, and an angle opposite to one of these sides equal to a given angle.*

DEFINITIONS.

44. A portion of a plane surface enclosed by straight lines is called a **plane rectilineal figure**.

The straight lines enclosing a plane rectilineal figure are called **sides**.

The sum of all the sides is called the **perimeter**.

The angles formed by adjacent sides are called the **angles** of the figure.

The points at which adjacent sides meet are called **vertices**.

A **diagonal** is a straight line joining any two vertices not extremities of the same side.

The quantity of the plane surface enclosed by the sides is called the **area**.

45. A **polygon** is a plane rectilineal figure which has more than four sides.

A **pentagon** is a polygon of *five* sides, a **hexagon** one of *six* sides, a **heptagon** one of *seven* sides, an **octagon** one of *eight* sides, etc.

46. A **quadrilateral** is a plane rectilineal figure which has four sides.

47. A **triangle** is a plane rectilineal figure which has three sides.

When any three straight lines in a plane intersect two and two, as in Fig. 92, they enclose a part of the plane, and thus form a triangle. Each point of intersection is a **vertex** of the triangle. The parts of the straight lines lying between the vertices two and two, are the **sides** of the triangle.

The angles formed by the sides of a triangle are called the **angles**—sometimes the **interior angles**—of the triangle.

An angle formed by any side and the prolongation of another side is called an **exterior angle** of a triangle. (See Fig. 91.)

It is often convenient to regard a triangle as standing upon one of its sides. Any side which may be selected is termed the **base** of the triangle. The point at which the other two sides meet is then called the **vertex** of the triangle.

The perpendicular distance from the vertex of a triangle to the base (produced if necessary) is called the **altitude** or **height** of the triangle.

A straight line drawn from any vertex of a triangle to the middle point of the opposite side is called a **median** of the triangle.

48. A triangle which has three sides equal is called an **equilateral triangle**.



FIG. 96.



FIG. 97.

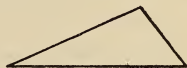


FIG. 98.

49. A triangle which has two sides equal is called an **isosceles triangle**.

The term *isosceles*, as here used, denotes the equality of two sides without reference to the third. Hence, an equilateral triangle may be regarded as isosceles when only two of its sides are considered.

50. A triangle which has three unequal sides is called a **scalene triangle**.

51. A triangle which has a right angle is called a **right-angled triangle**.

In a right-angled triangle the side opposite the right angle is called the **hypotenuse**.

52. A triangle which has an obtuse angle is called an **obtuse-angled triangle**.

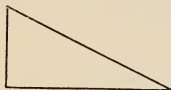


FIG. 99.



FIG. 100.



FIG. 101.

53. A triangle which has three acute angles is called an **acute-angled triangle**.

THEOREMS.

SECTION C.—TRIANGLES.

I.—Theorem 1.—(Euc. I. 4.)

If two triangles have two sides and the included angle of the one respectively equal to two sides and the included angle of the other, then the triangles are equal in all respects.

Given. That, in the two triangles ABC , DEF , the side AB is equal to the side DE , the side AC is equal to the side DF , and the included angle BAC is equal to the included angle EDF .

To be proved. That the base BC is equal to the base EF ; that the angle ABC is equal to the angle DEF , and the

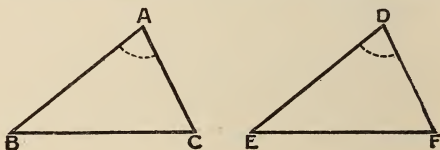


FIG. 102.

angle ACB is equal to the angle DFE ; also, that the triangles ABC , DEF are equal in area.

Proof. Suppose the triangle ABC to be applied to the triangle DEF so that the point A may be on the point D , and the straight line AB may fall along the straight line DE , then because AB is equal to DE , *Given.*

therefore the point B must fall on the point E .

(Equal straight lines can be made to coincide.)

And because AB lies along DE , and the angle BAC is equal to the angle EDF , *Given.*

therefore AC must fall along DF .

(Equal angles can be made to coincide.)

And because AC is equal to DF , *Given.*
therefore the point C must fall on the point F .

Now as B is on E , and C on F , BC must lie along EF ; for if not, two straight lines would enclose a space, which is impossible. *Ax.10.*

Hence, the base BC coincides with the base EF ;
therefore the base BC is equal to the base EF . *Ax.8.*

Also, the angles ABC , ACB coincide respectively with the angles DEF , DFE ;

therefore the angle ABC is equal to the angle DEF , and the angle ACB is equal to the angle DFE . *Ax.8.*

Also, the triangle ABC coincides with the triangle DEF ;
therefore the triangles ABC , DEF are equal in area.

Ax.8.

That is, the triangles ABC , DEF are equal in all respects.

Q.E.D.

1. AB is a given straight line which is bisected at D . Through D a straight line PQ is drawn perpendicular to AB . Show that every point in PQ is equidistant from the extremities of AB .

2. Show that the bisector of the vertical angle of an isosceles triangle will pass through the middle point of the base.

3. The straight lines drawn from the ends of the base of an isosceles triangle to the middle points of the opposite sides are equal.

II.—Theorem 2.—(Euc. I. 16.)

*If any side of a triangle is produced, then the exterior angle is greater than either of the interior opposite angles.**

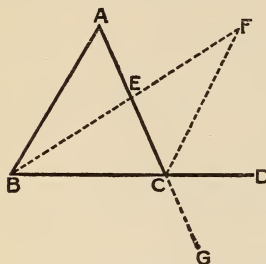


FIG. 103.

Given. That ABC is a triangle, of which the side BC is produced to D .

To be proved. That the exterior angle ACD is greater than either of the interior opposite angles BAC or ABC .

Construction. Let AC be bisected at E .†

Join BE . Produce BE to F , making EF equal to EB . Join CF .

Proof. Then in the triangles AEB , CEF , because

$$\left\{ \begin{array}{ll} AE \text{ is equal to } CE, & \text{Const.} \\ EB \text{ is equal to } EF, & \text{Const.} \\ \text{and the angle } AEB \text{ is equal to angle } CEF; & A.4-I.15. \end{array} \right.$$

therefore the triangles are equal in all respects. C.1—I.4.

Hence the angle EAB is equal to the angle ECF .

But the angle ECD is greater than the angle ECF , Ax.9.

*When one side of a triangle is produced one of the three interior angles is *adjacent* to the exterior angle, and the other two are *remote* from it. The two remote angles of the triangle are called *interior opposite* angles. Thus BAC and ABC in Fig. 103 are interior opposite angles.

†It has been shown that a straight line can have only one point of bisection, therefore only one straight line can be drawn from B to the middle point of AC . We need not here consider whether we can find the exact position of E on the diagram or not, as this has nothing whatever to do with the validity of the argument. In practical geometry the diagram is all-important, for the conclusion depends upon it; but in theoretical geometry it merely serves as an aid in following out a chain of reasoning.

therefore the angle ECD is greater than the angle EAB ;
that is, the angle ACD is greater than the angle BAC .

By producing AC to G , it can be shown in a similar manner that the angle BCG is greater than the angle ABC .

But the angle BCG is equal to the angle ACD ; *A.4—I.15.*
therefore the angle ACD is greater than the angle ABC .
Q.E.D.

III.—Theorem 3.—(Euc. I. 17.)

Any two angles of a triangle are together less than two right angles.

Given. That ABC is a triangle.

To be proved. That any two of its angles ABC , ACB are together less than two right angles.

Construction. Produce BC to D .

Proof. Write the proof.

1. In Fig. 103 join AF . Show that the triangles AEF , CEB are equal in all respects. State definitely in what respects they are equal.

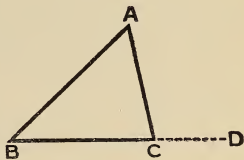


FIG. 104.

2. If a side of a triangle is produced both ways, the sum of the exterior angles thus formed is greater than two right angles.

3. Show that every right-angled triangle must have two acute angles.

4. *From a point without a straight line only one perpendicular to the line can be drawn.*

5. Prove Theorem 3 by joining the vertex to a point in the base.

6. ABC is a right-angled triangle of which the angle ACB is the right angle. Show that the straight line drawn through C perpendicular to AB must fall within the triangle.

7. In an obtuse-angled triangle, the perpendicular from the vertex of the obtuse angle to the opposite side falls within the triangle.

IV.—Theorem 4.—(Euc. I. 5.)

If two sides of a triangle are equal, then the angles opposite to these sides are equal.



FIG. 105.

Given. That ABC is an isosceles triangle having the side AB equal to the side AC .

To be proved. That the angle ABC is equal to the angle ACB .

Construction. Let AD be the bisector of the vertical angle BAC , and let AD meet the base BC at D .

Proof. Then in the triangles BAD , CAD , because

$$\left\{ \begin{array}{ll} AB \text{ is equal to } AC, & \text{Given.} \\ AD \text{ is common to both,} & \\ \text{and the angle } BAD \text{ is equal to the angle } CAD; & \text{Def. 21.} \end{array} \right.$$

therefore the triangles are equal in all respects. $C.1—I.4$.

Hence the angle ABC is equal to the angle ACB .

Q.E.D.

COROLLARY :

If a triangle is equilateral it is also equiangular.

V.—Theorem 5.—(Euc. I. 18.)

If two sides of a triangle are unequal, then the opposite angles are unequal, and the greater side has opposite to it the greater angle.

Given. That ABC is a triangle having the side AC greater than the side AB .

To be proved. That the angle ABC is greater than the angle ACB .

Construction. From AC the greater let AD be cut off equal to AB the less. Join BD .

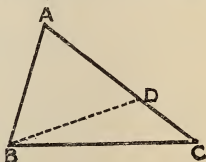


FIG. 106.

Proof. Because AB is equal to AD , *Const.*
therefore the angle ABD is equal to the angle ADB . *C.4—I.5.*

But the angle ADB is an exterior angle of the triangle BDC ,

therefore the angle ADB is greater than the angle ACB . *C.2—I.16.*

Therefore the angle ABD is greater than the angle ACB .

But the angle ABC is greater than the angle ABD ; *Ax.9.*
much more then is the angle ABC greater than the angle ACB . *Q.E.D.*

VI.—Theorem 6.—(Eucl. I. 6.)

If two angles of a triangle are equal, then the opposite sides are equal.

Given. That ABC is a triangle having the angle ABC equal to the angle ACB .



FIG. 107.

To be proved. That the side AC is equal to the side AB .

Proof.* Let us suppose that AC is not equal to AB . Then AC must be either greater than or less than AB . Now in either case the angle ABC must be unequal to the angle ACB ;

C.5—I.18.

but these angles are not unequal. *Given.*

Therefore AB is not unequal to AC ; that is, AB is equal to AC . *Q.E.D.*

COROLLARY :

If a triangle is equiangular it is also equilateral.

* If preferred, Euclid's proof may be substituted for the one here given. (See page 186.)

VII.—Theorem 7.—(Euc. I. 19.)

If two angles of a triangle are unequal, then the opposite sides are unequal, and the greater angle has opposite to it the greater side.

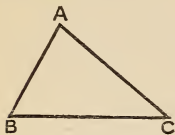


FIG. 108.

Given. That $\triangle ABC$ is a triangle having the angle $\angle ABC$ greater than the angle $\angle ACB$.

To be proved. That the side AC is greater than the side AB .

Proof. Now AC must be greater than AB , or equal to AB , or less than AB .

But AC is not equal to AB , for the angle $\angle ABC$ would then be equal to the angle $\angle ACB$,

C.4—I.5.

which is not the case.

Given.

Neither is AC less than AB , for the angle $\angle ABC$ would then be less than the angle $\angle ACB$,

C.5—I.18.

which is not the case.

Given.

Then because AC is neither equal to, nor less than AB , therefore AC is greater than AB .

Q.E.D.

1. No two angles of a scalene triangle can be equal.
2. In an obtuse-angled triangle the greatest side of the triangle is opposite the obtuse angle.
3. The side QR of the equilateral triangle PQR is produced to T ; show that PT is greater than PQ .
4. *The perpendicular is the shortest of all the straight lines that can be drawn from a given point to a given straight line.*
5. In a right-angled triangle the hypotenuse is the greatest side.
6. $\triangle ABC$ is a triangle of which the angle at B is a right angle. P and Q are points in BC and BC produced respectively. Prove that AQ is greater than AP .
7. In the triangle PQR let the bisector of the angle at P meet the side QR at S . Show that QS is less than PQ , and RS less than PR .
8. *Any straight line drawn from the vertex of a triangle to a point in the base is less than the greater of the two sides, or than either if they are equal.*

9. A straight line cannot cut the circumference of a circle at more than two points.

10. A straight line perpendicular to any diameter of a circle at its extremity is a tangent to the circle.

HINT.—Show that every point but one in the perpendicular must lie without the circle.

11. If one angle of a triangle is equal to the sum of the other two, then the triangle can be divided into two isosceles triangles.

12. State the relation which the last four theorems bear to one another.

VIII.—Theorem 8.—(Euc. I. 20.)

Any two sides of a triangle are together greater than the third side.

Given. That ABC is a triangle.

To be proved. That any two of its sides BA , AC are together greater than the third side CB .

Construction. Produce BA to D , making AD equal to AC . Join CD .

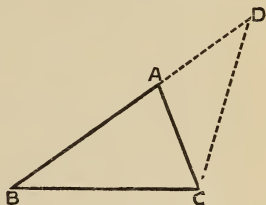


FIG. 109.

Proof. Then because AD is equal to AC ; Const.
therefore the angle ACD is equal to the angle ADC .

C.4—I.5.

But the angle BCD is greater than the angle ACD , *Ax. 9.*
therefore the angle BCD is greater than the angle BDC .

Then in the triangle BCD , because the angle BCD is greater than the angle BDC ,

therefore the side BD is greater than the side CB . *C.7—I.19.*

But BA , AC are together equal to BD ;
therefore BA , AC are together greater than CB .

It may be shown in a similar manner that AC , CB are together greater than BA ; also, that CB , BA are together greater than AC . Q.E.D.

1. Let the triangles ABC , DBC be on the same base and on the same side of it, the vertex of each being without the other. If AC and BD intersect, then their sum is greater than the sum of AB and DC .

2. The sum of the straight lines drawn from any point to the vertices of a quadrilateral is greater than half the perimeter.

3. Any side of a quadrilateral or polygon is less than the sum of all the other sides.

4. *The difference of any two sides of a triangle is less than the third side.*

IX.—The Principle of Continuity.

In proving the truth of a theorem we refer to a diagram representing the given conditions of the theorem. But this diagram is only one out of an unlimited number which would correspond to such conditions. Why does the consideration of one case enable us to say with certainty that the theorem is true in all possible cases?

Take for example Theorem 8. By proving that any two sides of the triangle ABC are together greater than the third side, we feel justified in making the sweeping assertion that in every triangle the sum of any two sides is greater than the third. The reason is simple, and may be stated briefly. It is this: the conditions made use of in proving the theorem as referred to the triangle ABC are such as *hold good in the case of every possible triangle*. This is but one illustration of what is known as the "*Principle of Continuity*"—one of the most important principles of mathematics. Give other illustrations.

X.—Theorem 9.—(Euc. I. 8.)

If two triangles have the three sides of the one respectively equal to the three sides of the other, then the triangles are equal in all respects.

Given. That in the triangles ABC , DEF , AB is equal to DE , AC is equal to DF , and BC is equal to EF .

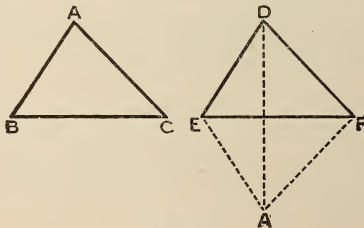


FIG. 110.

To be proved. That the triangles ABC , DEF are equal in all respects.

Proof. Let the triangle ABC be applied to the triangle DEF so that BC may coincide with its equal EF , B falling on E and C on F , and so that the vertex A may be at A' on the side of EF remote from D . Join $A'D$.

CASE I.—When $A'D$ crosses EF .

Because EA' is equal to ED , Given.
therefore the angle $EA'D$ is equal to the angle EDA' .
C.4—I.5.

Again because FA' is equal to FD . Given.
therefore the angle $FA'D$ is equal to the angle FDA' .
C.4—I.5.

Hence the whole angle $EA'F$ is equal to the whole angle EDF ,
Ax. 2.

that is, the angle BAC is equal to the angle EDF .

Therefore the triangles ABC , DEF are equal in all respects.
C.1—I.4.

Q.E.D.

CASE II.—When $A'D$ meets EF produced either way.

CASE III.—When $A'D$ passes through either extremity of EF .

Prove these cases.

1. If a straight line is drawn from the vertex of an isosceles triangle to the middle point of the base it is perpendicular to the base, and bisects the vertical angle.

2. Two isosceles triangles, ABC , DBC stand on the same base BC . Show that AD (produced if necessary) (i) bisects both vertical angles; (ii) bisects BC at right angles. Show that every point in AD is equidistant from B and C .

3. Two circles whose centres are P and Q intersect at X and Y . Show that PQ is perpendicular to XY , and passes through its middle point.

4. AB is a chord of the circle ABC . DE bisects AB at right angles, meeting the circumference at D and E . Prove that the centre of the circle is in DE . (Indirect proof.)

XI.—Theorem 10.—(Euc. I. 26.)

If two triangles have a side and the two adjacent angles of the one respectively equal to a side and the two adjacent angles of the other, then the triangles are equal in all respects.

Given. That ABC , DEF are two triangles, having the side BC equal to the side EF , the angle ABC equal to the angle DEF , and the angle ACB equal to the angle DFE .

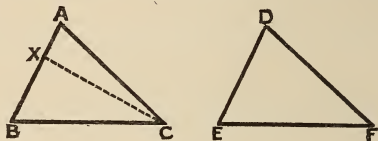


FIG. 111.

To be proved. That the triangles ABC , DEF are equal in all respects; that is, AB is equal to DE , AC is equal to DF , the angle BAC is equal to the angle EDF , and the area of the triangle ABC is equal to the area of the triangle DEF .

Now AB is greater than, or equal to, or less than DE . Suppose AB to be greater than DE .

Construction. Let BX be equal to ED . Join CX .

Proof. Then in the triangles XBC , DEF , because

$$\left\{ \begin{array}{ll} XB \text{ is equal to } DE, & \text{Const.} \\ BC \text{ is equal to } EF, & \text{Given.} \\ \text{and the angle } XBC \text{ is equal to the angle } DEF, & \text{Given.} \end{array} \right.$$

therefore these triangles are equal in all respects; *C.1.—I.4.*

hence, the angle XCB is equal to the angle DFE .

But the angle ACB is equal to the angle DFE , *Given.*

therefore the angle XCB is equal to the angle ACB : *Ax. 1.*

the part is thus equal to the whole, which is absurd. *Ax. 9.*

Therefore AB is not greater than DE .

Similarly it may be shown that AB is not less than DE ;
therefore AB is equal to DE .

Therefore the triangles ABC , DEF are equal in all respects; C.1—I.4.

so that the side AC is equal to the side DF , the angle BAC is equal to the angle EDF , and the area of the triangle ABC is equal to the area of the triangle DEF . Q.E.D.

XII.—Theorem 11.—(Euc I. 26.)

If two triangles have two angles of the one respectively equal to two angles of the other, and have also a side of one equal to a side of the other, these sides being opposite to a pair of equal angles; then the triangles are equal in all respects.

Given. That ABC , DEF are two triangles having the angles ABC , ACB equal to the angles DEF , DFE , respectively, and also the side AB equal to the side DE .

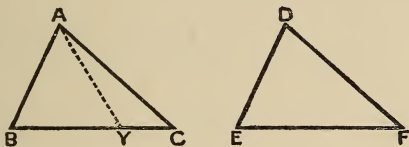


FIG. 112.

To be proved. That the triangles ABC , DEF are equal in all respects; that is, BC is equal to EF , AC is equal to DF , the angle BAC is equal to the angle EDF , and the area of the triangle ABC is equal to the area of the triangle DEF .

BC is greater than, or equal to, or less than EF . Suppose BC to be greater than EF .

Construction. Let BY be equal to EF . Join AY .

Write the proof.

XIII.—Theorem 12.

If two right-angled triangles have a side and the hypotenuse of the one respectively equal to a side and the hypotenuse of the other, then the triangles are equal in all respects.

Outline.—In the triangles ABC , DEF let the angles ABC , DEF be right angles, and let the sides AC , AB be respectively equal to the sides DF , DE : it is required to prove that the triangles are equal in all respects.

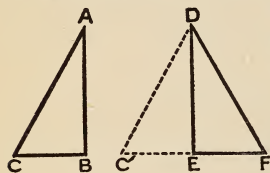


FIG. 113.

Place the triangle ABC so that AB may coincide with DE , C and F being on opposite sides of DE . Let C take the position of C' . $C'EF$ is a straight line. $DC'F$ is an isosceles triangle of which the angles at C' and F are equal. Hence the triangles ABC , DEF are equal in all respects.

Write the argument in full.

1. If the bisector of the vertical angle of a triangle is perpendicular to the base, the triangle is isosceles.
2. The straight line drawn from the vertex of an isosceles triangle perpendicular to the base bisects the vertical angle, and meets the base at its middle point.
3. If the opposite sides of a quadrilateral are equal, the diagonals bisect each other.
4. The perpendiculars drawn to the arms of an angle from any point in its bisector are equal.
5. If the perpendiculars drawn from a point to two intersecting straight lines are equal, then the point must be in the bisector of one of the angles included by these straight lines.
6. The perpendiculars to the sides of a triangle passing through the middle points of the sides meet at a point.
7. The bisectors of the angles of a triangle meet at a point.
8. State four different sets of given conditions by which we can prove that one triangle is identically equal to another.

PROBLEMS.

1. Demonstrate Fundamental Problems 2-4, 6-10, and 12-15, inclusive ; also, Fundamental Loci 1 and 2.

2. At a given point in a given straight line make an angle equal to (i) the complement of a given angle, (ii) the supplement of a given angle.

3. To make an angle equal to (i) the sum of two given angles, (ii) the difference of two given angles.

4. In a given straight line of unlimited length find two points equidistant from a given point.

5. In a given straight line of unlimited length find a point equidistant from two given points.

6. Describe an isosceles triangle having given :

- (i) The base and the height.
- (ii) One of the equal sides and the vertical angle.
- (iii) The base and one of the angles at the base.
- (iv) The base and the sum of the equal sides.
- (v) The altitude and the vertical angle.
- (vi) The altitude and the perimeter.

7. **ABC** is a given triangle. Construct in three different ways a triangle identically equal to the triangle **ABC**.

8. Describe a right-angled triangle having given :

- (i) The base and the height.
- (ii) The base and the hypotenuse.
- (iii) The base and the adjacent acute angle.
- (iv) The hypotenuse and the sum of the other sides.

9. Construct a triangle having given :

- (i) The base, one of the other sides and the perimeter.
- (ii) One angle, a side adjacent to it, and the difference of the other two sides.
- (iii) The base, the sum of the other two sides, and the vertical angle.

CHAPTER VIII.

PARALLEL STRAIGHT LINES.

Exercise XXXI.

1. Draw two straight lines AB and CD. Draw the straight line EF crossing them at the points G and H, as in Fig. 114.

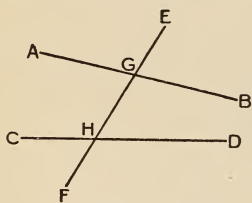


FIG. 114.

(i) Examine the angles formed by AB and EF with reference to the diagram as a whole. What pairs are formed towards the outside? What pairs are formed towards the inside?

(ii) Examine the angles formed by CD and EF.

(iii) Mark the four *exterior* angles, and also the four *interior* angles of the figure.

(iv) Note the position of the angle AGE with reference to the point of intersection of AB and EF. What angle has a corresponding position with reference to the point of intersection of CD and EF?

(v) What are the different pairs of *corresponding* angles?

[A straight line crossing two or more straight lines is called a *transversal*.]

2. Define *parallel straight lines*. Stretch two fine cords so that they may represent two parallel straight lines. What different conditions must the cords satisfy?

3. Without instruments test the parallelism of two edges of a sheet of foolscap.

4. Test the parallelism of the longest edges of your ruler.

5. Draw a transversal cutting two of the ruled lines on a sheet of foolscap. Examine a pair of alternate interior angles, and compare them as to size. State conclusion.

6. Draw two straight lines which would intersect if produced. Draw a transversal. Measure a pair of alternate interior angles. Compare them as to size. State conclusion.

7. Let two straight lines AB and CD intersect at E. In AE take any point P. Find the position of a straight line to be drawn through P parallel to CD. Draw the line.

8. Draw through a given point a straight line parallel to a given straight line.

(i) By means of ruler and set-square.

(ii) By means of protractor.

(iii) By means of compasses.

Show that in each case an angle is made equal to a given angle.

9. Draw a straight line crossing two parallel straight lines. Examine the diagram. (Use protractor.)

(i) What pairs of angles on the same side of the transversal are equal?

(ii) What pairs of angles on opposite sides of the transversal are equal?

(iii) What pairs of angles on the same side of the transversal are together equal to two right angles?

10. The straight line EF crosses the parallel straight lines AB and CD at G and H. The measure of the angle AGH is 50° . Calculate the measures of the other angles formed.

11. Draw a straight line parallel to the given straight line AB at a distance of $\frac{3}{4}$ in. from it. Test by superposition.

EXPLANATIONS.

It was seen in Chap. IV that there are three different ways in which two unlimited straight lines in a plane may be related to each other—they may *coincide*, they may *meet at a point*, or they may be *parallel*. This conclusion was based directly on the assumption that two straight lines cannot enclose a space. The relative position of two straight lines in a plane will now be dealt with, by considering the relation each bears to a third straight line which crosses them.

I.—Straight Lines Cut by a Transversal.

Let AB and CD be two straight lines crossed by the straight line EF, as in Fig. 115.

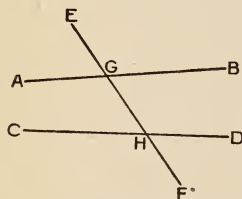


FIG. 115.

Then EF—called a *transversal*—forms with AB the four angles AGE, BGE, AGH and BGH; also, with CD, the four angles CHF, DHF, CHG and DHG. The pairs of angles thus formed are distinguished as follows:

- | | | |
|---|-----------|--|
| (i) Exterior angles | - - - - - | { AGE and BGE.
CHF and DHF. |
| (ii) Interior angles | - - - - - | { AGH and BGH.
CHG and DHG. |
| (iii) Corresponding angles | - - - | { AGE and CHG.
BGE and DHG.
AGH and CHF.
BGH and DHF. |
| (iv) Alternate interior angles
(on opposite sides of transversal.) | - - | { AGH and DHG.
BGH and CHG. |
| (v) Interior angles on the same side
(of transversal.) | | { AGH and CHG.
BGH and DHG. |

II.—Parallel Straight Lines.

Let AB and CD be two unlimited straight lines in a plane, and let EF cut them at the points P and Q . Suppose CD and EF to be fixed in position, and AB to be movable in the plane about the fixed point P .

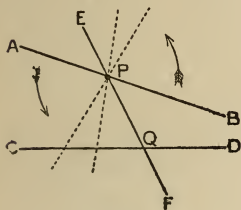


FIG. 116.

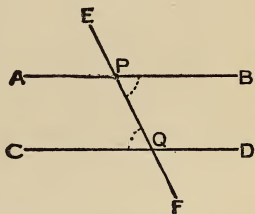


FIG. 117.

Let AB rotate about P , as indicated by the arrows in Fig. 116. Observe the following :

- (i) That AB will, in any number of different positions, meet CD .
- (ii) That in only one of the positions of AB will the alternate angles BPQ and CQP be equal to each other.

Let AB take the position in which it makes with EF the angle BPQ equal to the angle CQP , as in Fig. 117. In this position does AB meet CD , or is it parallel to CD ?

If AB and CD were to meet towards B and D , then a triangle would be formed as in Fig. 118, and the angles BPQ and CQP would be unequal. (C. 2—I. 16.) But these angles are equal; therefore we must conclude that AB and CD do not meet towards B and D . Similarly we can show that AB and CD do not meet towards A and C .

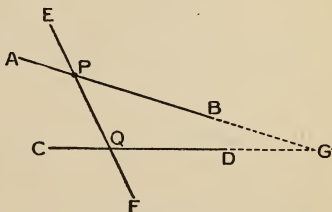


FIG. 118.

Now as AB and CD are two unlimited straight lines in a plane which do not meet, they are by definition *parallel* to each other.

It is thus shown that when AB is drawn through P so as to make the alternate angles BPQ and CQP equal, as in Fig. 117, then AB is parallel to CD . Hence through a given point *one* straight line can be drawn parallel to a given straight line. We shall next consider whether or not *more than one* straight line can be drawn through a given point parallel to a given straight line.

Let AB , as in Fig. 119, be drawn through the point P parallel to CD . Can any other straight line XY be drawn through P parallel to CD ?

This question we are unable to answer satisfactorily by any process of reasoning.* We, therefore, dispose of it by making an assumption which may be expressed thus :

Through a point there cannot be more than one straight line parallel to a given straight line.

As will be seen hereafter, our acceptance of this statement as an axiom opens up the way for a full consideration of parallel straight lines.

Show that a straight line crossing one of two parallel straight lines, must cross the other also, if the lines are produced far enough.

III.—Fundamental Problems.

16. *To draw a straight line parallel to a given straight line, and at a given distance from it.*

Let AB be the given straight line and d the given distance. It is required to draw a straight line parallel to AB and at the given distance d from it.

* For a discussion of this question see Henriëi's *Elementary Geometry*, Chap. V.

Take two points P and Q in AB . Draw PC and QD perpendicular to AB . With centre P and radius equal to d describe an arc cutting PC at X . With centre Q and radius the same as before describe an arc cutting QD at Y . Draw EF through the points X and Y . Then EF is parallel to AB , and at the given distance d from it.

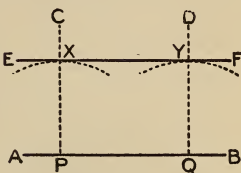


FIG. 120.

17. *Through a given point draw a straight line parallel to a given straight line.*

Let AB be the given straight line and P the given point. It is required to draw through P a straight line parallel to AB .

With centre P and any sufficient radius describe the arc CD cutting AB at C . With centre C and radius CP describe the arc PE , cutting AB at E . With centre C and radius equal

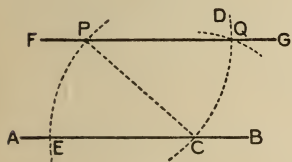


FIG. 121.

to EP (distance from E to P) describe an arc cutting the arc CD at Q . Draw the straight line FG through P and Q . Then FG is parallel to AB .

18. *To divide a given straight line into any number of equal segments.*

Let AB be the given straight line. It is required to divide it into, say, five equal segments.

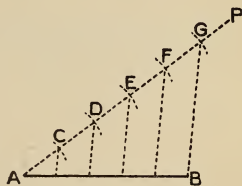


FIG. 122.

From A draw a straight line AP, forming any angle with AB. Along AP set off five equal segments AC, CD, DE, EF and FG. Join BG. Through C, D, E and F, draw straight lines parallel to BG. The straight lines thus drawn will cut AB as required.

IV.—Fundamental Loci.

3. *To find the locus of a point which is at a given distance from a given straight line.*

HINT.—Examine Fig. 120. Where must a point which is always at the distance d from AB lie?

4. *To find the locus of a point equidistant from two given parallel straight lines.*

5. *To find the locus of a point equidistant from two intersecting straight lines.*

HINT.—Examine Fig. 75. Show that every point in the bisector of the angle BAC satisfies this condition. Can any point not in the bisector satisfy it?

AXIOM.

11. Through a point there cannot be more than one straight line parallel to a given straight line.

THEOREMS.

SECTION D.—PARALLELS.

I.—Theorem 1.—(Euc. I. 27.)

If a straight line intersecting two other straight lines makes a pair of alternate angles equal, the two straight lines are parallel.

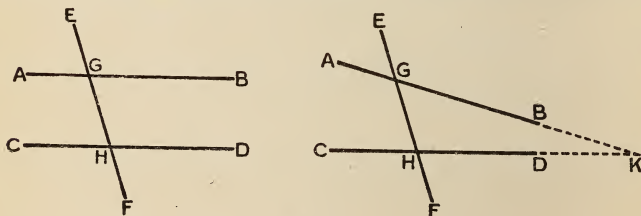


FIG. 123.

Given. That the straight line EF , cutting the straight lines AB , CD at the points G and H , makes the alternate angles AGH , GHD equal.

To be proved. That AB and CD are parallel.

Proof. Now AB , CD are either parallel, or not parallel. If they are not parallel, they will meet on being produced either towards B and D , or towards A and C .

If possible, let AB , CD produced meet towards B and D at K .

Then KGH is a triangle;
therefore the exterior angle AGH is greater than the interior opposite angle GHD . *C.2—I.16.*

But the angle AGH is equal to the angle GHD ; *Given.*
hence the angles AGH , GHD are both equal and unequal, which is impossible:

Therefore AB , CD do not meet when produced towards B and D .

In like manner it may be shown that they do not meet when produced towards A and C .

Therefore AB is parallel to CD .

Q.E.D.

COROLLARIES:

1. *If a straight line intersecting two other straight lines makes a pair of corresponding angles equal, the two straight lines are parallel.*

For, if the corresponding angles EGB , GHD are equal, then the alternate angles AGH , GHD are equal. Why?

2. *If a straight line intersecting two other straight lines makes a pair of interior angles on the same side of it together equal to two right angles, the two straight lines are parallel.*

For, if the interior angles BGH , GHD are together equal to two right angles, then the alternate angles AGH , GHD are equal. Why?

NOTE.—Theorem 1, and its two corollaries—the three simple tests of parallelism given by Euclid—are closely related to one another. When the truth of any one of them is established, that of the other two follows immediately from it.

1. If a transversal of two straight lines makes one pair of alternate interior angles equal, then

(i) The other pair of alternate interior angles are equal.

(ii) Each pair of corresponding angles are equal.

(iii) Each pair of interior angles on the same side of the transversal are supplementary.

(iv) Each pair of exterior angles on the same side of the transversal are supplementary.

2. Deduce Cor. 2 from C. 3—I. 17.

3. All straight lines perpendicular to the same straight line are parallel.

II.—Theorem 2.—(Euc. I. 29.)

If two parallel straight lines are intersected by a third straight line, the alternate angles are equal.

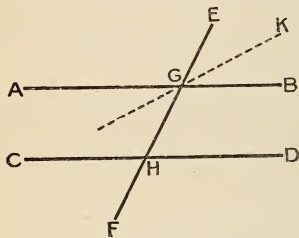


FIG. 124.

Given. That the two parallel straight lines AB and CD are intersected by the straight line EF at the points G and H .

To be proved. That the alternate angles BGH and GHC are equal.

Proof. The angles BGH , GHC are either equal or unequal. Suppose that they are unequal, and let the angle KGH —formed by drawing through G the straight line GK —be equal to the angle GHC .

Then KG must be parallel to CD .

D.1—I.27.

But AB is parallel to CD ;

Given.

therefore two straight lines are drawn through G , both of which are parallel to CD , which is impossible.

Ax.11.

Hence the supposition that the alternate angles BGH , GHC are unequal is absurd ; that is, they are equal.

Q.E.D.

COROLLARIES :

1. *If two parallel straight lines are intersected by a third straight line, the corresponding angles are equal.*

For the alternate angles BGH , GHC are equal ; hence the corresponding angles AGE , GHC are equal. Why?

2. *If two parallel straight lines are intersected by a third straight line, the interior angles on the same side are together equal to two right angles.*

For the alternate angles BGH , GHC are equal ; hence the interior angles AGH , GHC are together equal to two right angles. Why?

1. Two intersecting straight lines cannot be both parallel to the same straight line.

2. If two parallel straight lines are intersected by a third straight line, then the exterior angles on the same side of the line are supplementary.

3. AB , CD are two parallel straight lines cut by the transversal EF at the points E and F . Through O , the point of bisection of EF , the straight line XY is drawn meeting AB , CD at X and Y . Show that OX is equal to OY , and EX to FY .

4. *Straight lines which are parallel to the same straight lines are parallel to one another.*

5. *A straight line perpendicular to one of any number of parallel straight lines is perpendicular to all of these lines.*

6. *If the arms of one angle are respectively parallel to the arms of another, these angles are either equal or supplementary.*

7. If a transversal of two straight lines makes the interior angles on one side of it together less than two right angles, the two straight lines cannot be parallel.

III.—Theorem 3.—(Euc. I. 32.)

Any exterior angle of a triangle is equal to the sum of the two interior opposite angles ; also, the sum of the three interior angles of a triangle is equal to two right angles.*

Given. That ABC is a triangle and ACD an exterior angle of it.

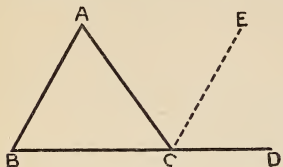


FIG. 125.

To be proved. (i) That the exterior angle ACD is equal to the sum of the two interior opposite angles CAB , ABC .

(ii) That the sum of the three interior angles ABC , BCA , CAB is equal to two right angles.

Construction. Through C draw CE parallel to BA .

Proof. (i) Because AC meets the parallels BA , CE , therefore the angle ACE is equal to the alternate angle CAB . D.2—I.29.

Again, because BD meets the parallels BA , CE , therefore the angle ECD is equal to the corresponding angle ABC . D.2—I.29.

therefore the whole exterior angle ACD is equal to the sum of the two interior opposite angles CAB , ABC .

(ii) To each of these angles add the angle BCA ; then the sum of the angles BCA , ACD is equal to the sum of the angles BCA , CAB , ABC .

But the sum of the angles BCA , ACD is equal to two right angles; A.2—I.13.

therefore the sum of the angles BCA , CAB , ABC is equal to two right angles. Q.E.D.

1. If two triangles have two angles of the one equal to two angles of the other, respectively, their third angles are equal.

2. If two angles of a triangle are complementary, the triangle is right-angled.

3. In an equilateral triangle each of the angles is two-thirds of a right angle.

4. The angles of a quadrilateral are together equal to four right angles.

5. Show that a right-angled triangle can be divided into two isosceles

triangles; also, that the middle point of the hypotenuse is equidistant from the three vertices.

6. The bisector of the exterior vertical angle of an isosceles triangle is parallel to the base. State the converse of this theorem.

7. The bisectors of the angles at the base of an isosceles form an angle equal to an exterior angle at the base.

8. Prove D. 3 by drawing through **A** a straight line parallel to **BC**.

9. The bisectors of the angles **B** and **C** of the triangle **ABC** meet at **D**. Through **D** straight lines **DE** and **DF** are drawn parallel to **AB** and **AC**, meeting **BC** at **E** and **F**. Show that the perimeter of the triangle **DEF** is equal to **BC**; also, that the angles of the triangle **DEF** are respectively equal to those of the triangle **ABC**.

IV.—Theorem 4.—(Euc. III. 31.)

An angle in a semicircle is a right angle.

Outline.—Let **ABC** be an angle in the semicircle **ACD** of which **O** is the centre. Join **BO**. The angle **OBA** is equal to the angle **OAB**, and the angle **OBC** to the angle **OCB**. The whole angle **ABC** is equal to the angles **CAB** and **ACB** together. Hence the conclusion.

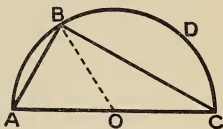


FIG. 126.

Write the argument in full.

PROBLEMS.

1. Demonstrate Fundamental Problems 5, 11, 16 and 17; also, Fundamental Loci 3, 4 and 5.

2. Construct a right-angled triangle having given :

(i) The hypotenuse, and one of the acute angles.

(ii) The hypotenuse and one side.

3. Construct a triangle having given :

(i) The altitude, the vertical angle, and one of the angles at the base.

(ii) The base, the vertical angle, and one of the angles at the base.

(iii) The base, one of the sides, and the altitude.

(iv) The base, the altitude, and one angle at the base.

(v) The perimeter and two angles. (See Ex. 9 above.)

CHAPTER IX.

QUADRILATERALS.

Exercise XXXII.

1. Draw three four-sided rectilinear figures, differing in shape. What characteristics are common to all the figures?

Define *quadrilateral*.

2. Draw a quadrilateral ABCD as in Fig. 127. Join AC.

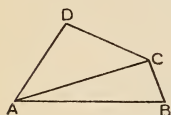


FIG. 127.

Compare the sum of the angles of the quadrilateral with that of the angles of the triangles ABC and ADC. Find the sum of the angles of ABCD.

Define *diagonal* of quadrilateral.

3. Produce AB, BC, CD, and DA in No. 2 in the same order. Find the sum of the exterior angles thus formed.

4. The sum of three angles of a quadrilateral is 300° . What is the measure of the other angle? Illustrate.

5. Draw quadrilaterals having (a) no two sides parallel, (b) only two sides parallel, (c) opposite sides parallel.

Define *trapezoid*, *parallelogram*.

6. Place two straight lines AB and AD 2 in. and 3 in. long, respectively, so that the included angle BAD may be 60° . Complete the parallelogram ABCD. Make measurements of diagram. What sides and angles are equal?

7. Draw any parallelogram ABCD. Draw diagonals AC and BD intersecting at O. Compare with compasses AO and CO; also, BO and DO. State conclusion.

Deduce a method of drawing a parallelogram when the lengths of the diagonals are given.

8. Draw a parallelogram whose diagonals include an angle of 50° , their lengths being $2\frac{1}{2}$ in. and 3 in.

9. Draw three parallelograms, each having a pair of adjacent sides $1\frac{1}{2}$ in. and 2 in. long :

(i) Let the included angle be acute.

(ii) Let the included angle be obtuse.

(iii) Let the included angle be a right angle.

10. Draw, as in No. 9, three parallelograms, each having a pair of adjacent sides 2 in. long.

Define *rectangle rhombus, square*.

11. Let ABCD be a square, whose side is 3 in. long. Draw a diagonal. Make such measurements of the triangles thus formed as will enable you to say whether they can be made to coincide or not.

12. Draw a parallelogram having its adjacent sides 3 in. and 4 in. long, and the angle included by these sides $\frac{3}{5}$ of a right angle. Calculate in degrees :

(i) The measure of each of its angles.

(ii) The measure of the sum of each pair of opposite angles.

(iii) The measure of each of its exterior angles.

EXPLANATIONS.

In Chapter VII we learned that a quadrilateral is a plane figure bounded by four straight lines. We shall now consider some of the forms which the quadrilateral may take, and also some of its properties.

I.—Kinds of Quadrilaterals.

Let one pair of unlimited straight lines, AB and CD , cross another pair, EF and GH , forming the quadrilateral $KLMN$, as in Fig. 128.

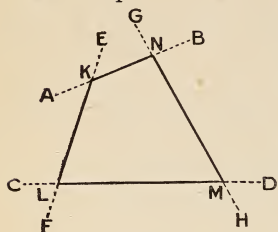


FIG. 128.

By examining the diagram we find that one pair of opposite sides, LK and MN , meet on being produced towards K and N ; also, that the other pair of opposite sides, ML and NK , meet on being produced towards L and K .

Suppose AB (Fig. 128) to rotate in the plane about the point K until it becomes parallel to CD , then the quadrilateral $KLMN$ will take the form shown in Fig. 129. As the quadrilateral $KLMN$ has now *one pair of opposite sides parallel*, it is called a **trapezoid**.

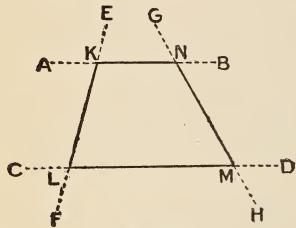


FIG. 129.

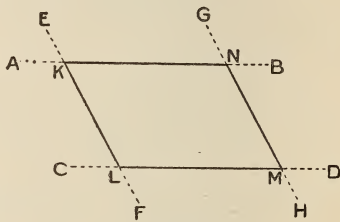


FIG. 130.

Again, suppose EF (Fig. 129) to rotate in the plane about the point L until it becomes parallel to GH , as in Fig. 130. The quadrilateral $KLMN$, having in this case *both pairs of opposite sides parallel*, is called a **parallelogram**.

The learner will observe that the trapezoid and the parallelogram are special forms of the quadrilateral.

II.—Angles of a Quadrilateral.

The sum of the exterior angles of a quadrilateral formed by producing its sides in the same order, is equal to a perigon, or four right angles.

Prove this practically by causing a straight-edge to turn successively through the exterior angles of a quadrilateral.*

Deduce the following :

The sum of the interior angles of a quadrilateral is equal to a perigon, or four right angles.

Show that this statement is true by dividing the quadrilateral into two triangles.

III.—Fundamental Problems.

19. *To construct a parallelogram having two of its sides respectively equal to two given straight lines, and the angle included by these sides equal to a given angle.*

ANALYSIS.—Draw any parallelogram according to definition. Examine the diagram and state the relations between its sides, angles, etc., that must always hold good. What changes must be made in the figure in order that it may conform to the requirements of the problem?

Let AB and CD be the given straight lines and E the given angle.

It is required to construct a parallelogram having two sides respectively equal to AB and CD, and the included angle equal to E.

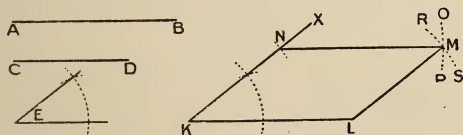


FIG. 131.

Draw the straight line KL equal to AB. At K make the angle LKX equal to E. From KX cut off KN equal to CD. With centre N and radius equal to AB describe the arc OP. With centre L and radius equal to CD describe the arc RS.

* An illustration of this form of proof as applied to the triangle, is given on page 95.

Let M be the point at which the arcs intersect. Join NM and LM . Then $KLMN$ is the parallelogram required.

Examine the diagram. How is the length of LM obtained? Of MN ? Upon what property of the parallelogram does this construction depend?

20. *To construct a rhombus, when one side and one angle are given.*

Draw a rhombus and explain the construction; afterwards draw the rhombus required.

21. *To construct a rectangle when two sides are given.*

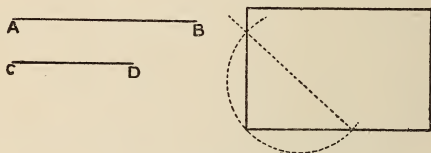


FIG. 132.

State what is given and what is required to be done. Examine the diagram, letter it, and explain its construction. Show that it conforms to the requirements of the definition of rectangle. Draw two rectangles.

22. *To construct a square on a given straight line.*

DEFINITIONS.

54. A quadrilateral which has only one pair of opposite sides parallel is called a **trapezoid**.

55. A quadrilateral whose opposite sides are parallel is called a **parallelogram**.

Any side of a parallelogram may be regarded as the **base**.

The **altitude**, or **height** of a parallelogram is the perpendicular distance between the base and the opposite side.

56. A parallelogram which has a right angle is called a **rectangle**.

As the sum of any two adjacent angles of a parallelogram is equal to two right angles, it follows that all the angles of a rectangle are right angles.

57. A parallelogram which has two adjacent sides equal is called a **rhombus**.

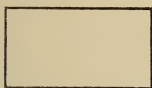


FIG. 133.



FIG. 134.



FIG. 135.

The opposite sides of a parallelogram being equal, it follows that all the sides of a rhombus are equal.

58. A rectangle which has two adjacent sides equal is called a **square**.

It follows from this definition and what has been stated above that all the sides of a square are equal, and all its angles are right angles.

THEOREMS.

SECTION E.—PARALLELOGRAMS.

I.—Theorem 1.—(Euc. I. 33.)

If two sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.

Given. That $ABDC$ is a quadrilateral having the sides AB , CD equal and parallel.

To be proved. That $ABDC$ is a parallelogram.

Construction. Draw a diagonal, BC .

Proof. Because BC meets the parallels AB , CD , therefore the angle ABC is equal to the alternate angle DCB .

D.2—I.29.

Then in the triangles ABC , DCB , because

$$\left\{ \begin{array}{l} AB \text{ is equal to } CD, \\ BC \text{ is common to both,} \\ \text{and the angle } ABC \text{ is equal to the angle } DCB; \end{array} \right.$$

therefore the angle ACB is equal to the angle DBC . *C.1—I.4.*

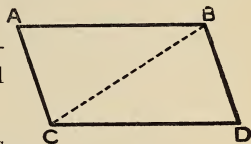


FIG. 136.

Because CB meets AC and BD , making the alternate angles ACB , DBC equal,

therefore AC is parallel to BD .

D.1—I.27.

Therefore $ABDC$ is a parallelogram.

Def.55.

Q.E.D.

II.—Theorem 2.—(Euc. I. 34.)

The opposite sides and angles of a parallelogram are equal, and each diagonal bisects it.

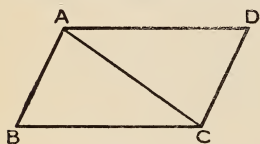


FIG. 137.

Given. That $ABCD$ is a parallelogram of which AC is a diagonal.

To be proved. That the opposite sides and angles of $ABCD$ are equal, and AC bisects it.

Proof. Because AB and DC are parallel and AC meets them,

therefore the alternate angles BAC , DCA are equal. *D.2—I.29.*

Because AD and BC are parallel and AC meets them, therefore the alternate angles BCA , DAC are equal. *D.2—I.29.*

Then in the triangles BAC , DCA , because

$\left\{ \begin{array}{l} \text{the angle } BAC \text{ is equal to the angle } DCA, \\ \text{the angle } BCA \text{ is equal to the angle } DAC, \\ \text{and the side } AC \text{ is common to both;} \end{array} \right.$

therefore the triangles BAC , DCA are equal in all respects. *C.10—I.26.*

That is, AB is equal to CD , CB to AD , the angle ABC to the angle CDA , and the area of the triangle BAC to the area of the triangle DCA .

Again, because the angle BAC is equal to the angle DCA , and the angle CAD is equal to the angle ACB ,

therefore the angle BAD is equal to the angle DCB . *Ax.9.*

Q.E.D.

1. *The straight lines joining the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel.*

2. The straight lines joining the extremities of two equal and parallel straight lines towards opposite parts are bisected at their point of intersection.

3. If the distances of the points **P** and **Q** from the straight line **AB** are equal, **AB** is either parallel to **PQ** or bisects **PQ**.

4. If a parallelogram has two adjacent sides equal, then all its sides are equal.

5. If a parallelogram has one right angle, then all its angles are right angles.

6. *If a quadrilateral has its opposite sides equal, it is a parallelogram.*

7. *The diagonals of a parallelogram bisect each other.*

PROBLEMS.

1. Demonstrate Fundamental Problems 16–22, inclusive.

2. To construct a square when a diagonal is given.

HINT.—Draw any square. How are its diagonals related to each other? Observe that the solution of the problem consists in finding two vertices of the required square.

3. To construct a rectangle, when one side and a diagonal are given.

HINT.—Two vertices can at once be found; for the distance between them is given. In what line must the other two vertices lie? (See D. 4—III. 31. Solve by the *Method of Loci*.)

4. To construct a rectangle, when a diagonal and the angle formed by it and one side are given. (*Method of Loci*.)

5. To construct a rhombus, when one diagonal and one side are given.

6. To construct a rhombus when both diagonals are given.

7. To construct a parallelogram, when both diagonals and the angle formed by them, are given.

HINT.—Suppose two of the vertices to be found, what is the locus of the other two? How is their exact position in that locus determined?

CHAPTER X.

AREAS.

Exercise XXXIII.

1. Mark out a figure equal in all respects to the face of this post card. What test of equality do you apply?

Draw a second figure equivalent to the first.

2. Draw two equal straight lines AB and CD. On each describe a square. Show that the squares are equal.

3. Draw a rectangle whose measure is 8 when the unit is 1 sq. in. Draw a second rectangle equivalent to the first.

4. Lay out on the ground a rectangle whose height is 10 ft. and area 120 sq. ft.

NOTE.—In laying out a right angle on the ground place on it three sticks 6 ft., 8 ft. and 10 ft. long, respectively, in the form of a triangle. The angle opposite to the side represented by the longest stick will be a right angle.

5. Let ABCD be a parallelogram. Produce CD to F, and draw BE and AF perpendicular to CF, as in Fig. 138. Show that the rectangle ABEF is equivalent to the parallelogram ABCD.

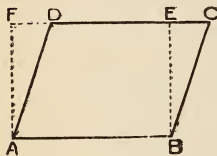


FIG. 138.

6. What is the area of a parallelogram whose base and height are 10 ft. and 6 ft. respectively? Draw to a scale of $\frac{1}{4}$

in. per foot.

7. How do we find the area of any parallelogram when the base and height are given?

Exercise XXXIV.

1. On AB describe any rectangle ABCD. Draw the diagonal AC. Compare the area of the triangle ABC with that of the rectangle.

2. On the ground lay out a right-angled triangle whose base is 10 ft. long, and whose height is 6 ft. Find its area. Explain by reference to a diagram drawn to a scale of $\frac{1}{4}$ in. per foot.

3. On BC describe any rectangle ABCD, as in Fig. 139. In AD take any point P. Join BP, and CP. Show that the area of the triangle BPC is half that of the rectangle ABCD. Draw two other triangles equal in area to the triangle BPC.

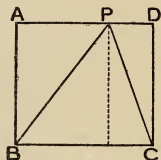


FIG. 139.

4. What is the area of a triangle of which the base is 13 ft. and height 10 ft.? Draw to a scale of $\frac{1}{4}$ in. per foot, giving the triangle two different forms.

5. Find the area of the trapezoid, as in Fig. 140. Draw to a scale of $\frac{1}{8}$ in. per foot.

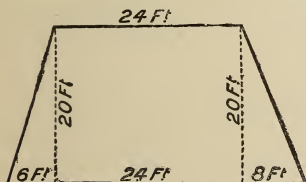


FIG. 140.

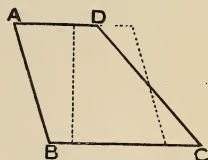


FIG. 141.

6. The dotted lines in Fig. 141 indicate one way of transforming the trapezoid ABCD into an equivalent rectangle. Explain fully. Devise another way of doing it.

7. One of the diagonals of a quadrilateral is 5 ft. long and the perpendiculars on it from the opposite vertices are $1\frac{1}{2}$ ft. and 2 ft. Draw on the blackboard and find the area. Draw on paper to a scale of $\frac{1}{2}$ in. per foot, giving the figure three different forms.

8. Draw a triangle ABC having the angle at B a right angle. On AB describe the square ABDE, as in Fig. 142. On BC describe a square, cut it out, and give it the position

of DFGH. By cutting the figure ABFGHE into parts and rearranging them, show that the squares on AB and BC are together equal in area to the square on AC.

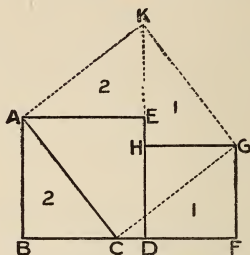


FIG. 142.

9. A rectangular field is 20 rods long and 15 rods wide. Find the area of the square described on its diagonal. Find the length of the diagonal. Illustrate by a diagram drawn to a scale of $\frac{1}{8}$ in. per rod.*

EXPLANATIONS.

I.—Area of a Rectangle.

To measure any surface we select a definite quantity of surface as the unit, and then we find how many times the unit-quantity is contained in the whole quantity of surface to be measured.

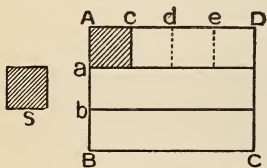


FIG. 143.

Let ABCD be a rectangle 4 units long and 3 units broad, and let the unit-quantity of surface by which the rectangle is to be measured be a square S whose side is 1 unit of length.

Divide AB into unit lengths at a and b . Through these points draw straight lines parallel to AD. The rectangle ABCD is thus divided into three equal rectangles. Observe that there are as many rectangles as there are units of length in AB.

* For additional exercises see High School Arithmetic, pp. 79-82.

Again, let AD be divided into unit lengths at c , d , and e . Through these points draw straight lines dividing the rectangle Da into squares. Observe that these four squares are each equal to S .

We thus see that the area of the whole rectangle $ABCD$ is 3 times 4 units of surface, or 12 units of surface.

Hence, *the number of units of surface in a rectangle is equal to the product of the numbers denoting the measures of its length and breadth, respectively.*

II.—Equivalent Figures.

Draw any rectangle $ABCD$, as in Fig. 144. In AD take any point E and join BE .

If from $ABCD$ the triangle BAE were cut off and given the new position CDF as in the diagram, then the figure $EBCF$ would be equal to the rectangle $ABCD$, for the figures would be successively made up of the same parts arranged in two different ways. Further, if the figure $EBCF$ were examined it would be found to be a parallelogram. We thus see how a rectangle may be transformed into an equivalent parallelogram.

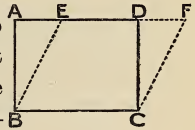


FIG. 144.

When the boundaries of two figures can be made to coincide with each other at all points, the figures are equal in area and similar in form. In applying the superposition test of equality to such figures, we suppose one of them to be moved without change of form and placed upon the other.

Axioms 8 and 9 taken together furnish us with a test applicable to certain figures which cannot be made to coincide without undergoing change of form.

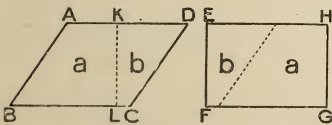


FIG. 145.

Let $ABCD$ and $EFGH$, as in Fig. 145, be two figures whose areas are to be compared.

(i) Divide $ABCD$ into two parts, as indicated by the dotted line KL . Then the area of the whole figure is equal to the sum of the areas of the parts. (Ax. 9.) Observe that the arrangement of the parts does not in any way affect the truth of this statement.

(ii) Let us now suppose that by rearranging the parts of the figure $ABCD$ it will coincide with the figure $EFGH$. Then the area of $ABCD$ is equal to the area of $EFGH$. (Ax. 8.)

The equality of the areas of two figures is frequently established in another way. Each of the figures is compared with a third. If their areas bear the same relation to that of the third then the two figures are equal in area.

III.—Fundamental Problems.

23. *To reduce a given polygon to an equivalent triangle.*

Let $ABCDE$ be the given polygon: it is required to reduce the polygon $ABCDE$ to an equivalent triangle.

Join AD . Through E draw EF parallel to AC meeting CD produced at F . Join AF . Then the quadrilateral $ABCF$ is equivalent to the pentagon $ABCDE$.

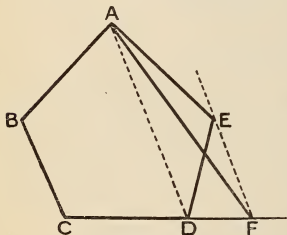


FIG. 146.

In a similar way the quadrilateral $ABCF$ may be reduced to an equivalent triangle.

Complete the construction.

Reduce a hexagon to an equivalent triangle.

24. *To construct a parallelogram that shall be equal in area to a given triangle, and have one of its angles equal to a given angle.*

Let ABC be the given triangle, and D the given angle: it is required to construct a parallelogram equivalent to the triangle ABC , and having one of its angles equal to D .

Bisect BC at E . At E in EC make the angle CEF equal to the given angle D . Through C draw CG parallel to EF , and through A draw AFG parallel to EC . Then $FECG$ is the parallelogram required.

Construct a parallelogram equal in area to the given pentagon $ABCDE$.

25. *On a given base to construct a parallelogram equal to a given triangle, and having an angle equal to a given angle.*

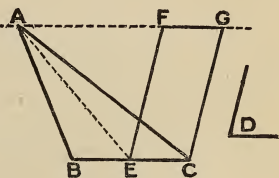


FIG. 147.

Let AB be the given straight line, C the given triangle, and D the given angle.

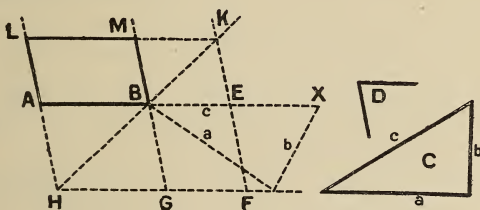


FIG. 148.

It is required to construct on AB a parallelogram equal to the given triangle C , and having an angle equal to the given angle D .

Produce AB to X , making BX equal to one of the sides c of the given triangle C . On BX describe a triangle equal in all respects to the triangle C . Construct the parallelogram $FEBG$ equivalent to the triangle thus described, and having the angle EBG equal to the given angle D . Produce FE , FG , GB . Through A draw HAL parallel to EF or GB , meeting FG produced at H . Draw HB and produce it to meet FE produced at K . Through K draw KL parallel to FH or EA , meeting GB produced at M , and HL at L .

Then $LABM$ is the parallelogram required.

Suggest another method of constructing the parallelogram FEBG. Can a different parallelogram be drawn to serve the same purpose in the construction? Illustrate.

On a given base construct a rectangle equal in area to a given rectangle.

NOTE.—Each of the three preceding Fundamental Problems serves an important purpose in the transformation of figures. Observe the way in which they are related to one another :

(i) No. 23.—A given *polygon having any number of sides* is converted into an equivalent polygon having one side fewer, and so on, until finally an equivalent *triangle* is obtained.

(ii) No. 24.—A given *triangle* is converted into an equivalent *parallelogram having one of its angles equal to a given angle*.

(iii) No. 25.—A given *triangle* is converted into an equivalent parallelogram having not only *one of its angles equal to a given angle*, but also *one of its sides equal to a given straight line*.

It is now evident that by means of Nos. 23 and 24 we can transform any polygon into an equivalent rectangle, and thus find the area of the polygon. No. 25 enables us further to make one of the sides of the rectangle of any length we please.

[Notice that the sequence here given is not quite complete ; for we cannot yet convert a polygon into an equivalent square.]

THEOREMS.

SECTION F.—AREAS.

I.—Theorem 1.—(Euc. I. 35.)

Parallelograms on the same base and between the same parallels are equal in area.

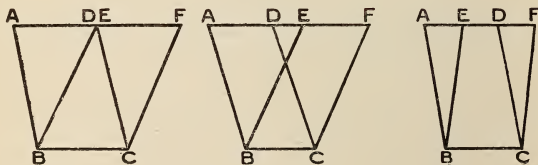


FIG. 149.

Given. That $ABCD$, $EBCF$ are parallelograms on the same base BC and between the same parallels AF , BC .

To be proved. That the parallelograms $ABCD$, $EBCF$ are equal in area.

Proof. Because $ABCD$ is a parallelogram, therefore AB is equal to DC . *E.2—I.34.*

Because AB is parallel to DC and AF meets them, therefore the angle BAE is equal to the angle CDF . *D.2—I.29.*

Again, because BE is parallel to CF and AF meets them, therefore the angle BEA is equal to the angle CFD . *D.2—I.29.*

Then in the triangles ABE , DCF , because

$$\left\{ \begin{array}{l} AB \text{ is equal to } DC, \\ \text{the angle } BAE \text{ to the angle } CDF, \\ \text{and the angle } BEA \text{ to the angle } CFD; \end{array} \right.$$

therefore the triangles are equal in all respects. *C.11—I.26.*

Hence the quadrilateral $ABCF$ diminished by the triangle ABE is equal to the same quadrilateral diminished by the triangle DCF . *Ax. 3.*

That is, the area of the parallelogram $ABCD$ is equal to that of the parallelogram $EBCF$. *Q.E.D.*

NOTE.—This theorem may be enunciated thus :

Parallelograms on the same base, and of equal altitudes, are equal in area.

Observe that the parallelograms $ABCD$ and $EBCF$ are not proved equal in all respects.

1. *Two parallelograms having two sides and the included angle of the one equal to two sides and the included angle of the other, respectively, are identically equal.* (Superposition.)

2. *The area of a parallelogram is equal to that of the rectangle whose base and altitude are equal to those of the parallelogram.*

3. *Parallelograms on equal bases and of equal altitudes are equal in area.*

4. State the converse of *F.1—I.35*, and prove it to be true.

II.—Theorem 2.—(Euc. I. 37.)

Triangles on the same base and between the same parallels are equal in area.

Given. That the triangles ABC , DBC are on the same base BC and between the same parallels AD , BC .

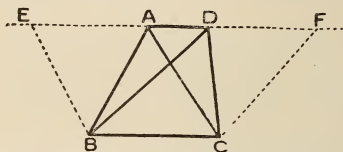


FIG. 150.

To be proved. That the triangles ABC , DBC are equal in area.

Construction. Through B let BE be drawn parallel to CA , meeting DA produced in E ; also, through C let CF be drawn parallel to BD , meeting AD produced in F .

Proof. Then because $EBCA$, $DBCF$ are parallelograms on the same base BC and between the same parallels BC , EF , therefore they are equal in area. *F.1—I.35.*

But the triangle ABC is half of the parallelogram $EBCA$;

E.2—I.34.

also, the triangle DBC is half of the parallelogram $DBCF$;

E.2—I.34.

therefore the triangle ABC is equal to the triangle DBC .

Ax.7.

Q.E.D.

*NOTE.—This theorem may be enunciated thus :

Triangles on the same base, and of equal altitudes, are equal in area.

Observe that the triangles ABC , DBC are *not* proved equal in all respects.

1. Triangles on equal bases and of equal altitudes are equal in area.
2. A straight line joins the middle points of two sides of a triangle. Show that

- (i) It is equal to half the third side and parallel to it.
- (ii) It divides the triangle into two parts, one of which is a third of the other.

III.—Theorem 3.—(Euc. I. 41.)

If a parallelogram and a triangle are on the same base and between the same parallels, the parallelogram is double of the triangle.

Given. That the parallelogram $ABCD$ and the triangle EBC are on the same base BC , and between the same parallels AE , BC .

To be proved. That the parallelogram $ABCD$ is double of the triangle EBC .

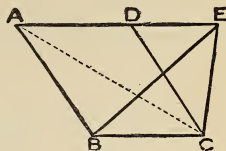


FIG. 151.

Construction. Join AC .

Proof. Then the area of the triangle ABC is equal to that of the triangle EBC , for they are on the same base BC , and between the same parallels AE , BC . F.2—I.37.

But the parallelogram $ABCD$ is double of the triangle ABC , for AC bisects the parallelogram. E.2—I.34.

Hence the parallelogram $ABCD$ is also double of the triangle EBC . Q.E.D.

1. Let $ABCD$ be any parallelogram. In AB and BC take any points P and Q , respectively. Show that the triangles PCD and QAD are equal in area.

2. P is any point in the plane of the parallelogram $ABCD$ between the parallels AB and DC . Show that the sum of the triangles PAB and PDC is equal to half the parallelogram.

3. Through the extremities of each diagonal of a quadrilateral straight lines are drawn parallel to the other diagonal. Prove that the figure

thus formed is a parallelogram whose area is double that of the given quadrilateral.

4. *If a parallelogram and a triangle are on equal bases and between the same parallels, the parallelogram is double of the triangle.*

IV.—Theorem 4.—(Euc. I. 47.)

In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Given. That $\triangle ABC$ is a right-angled triangle having the right angle BAC .

To be proved. That the square on BC is equal to the sum of the squares on BA , AC .

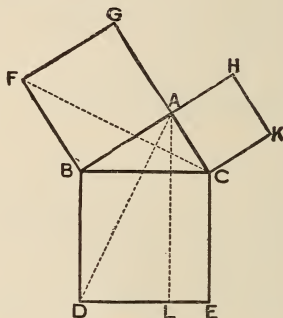


FIG. 152.

Construction. On BC , BA , AC , describe the squares $BDEC$, $BAGF$, $ACKH$ respectively. Through A draw AL parallel to BD or CE , meeting DE at L . Join AD , FC .

Proof. Because the angles BAC , BAG are right angles, therefore CAG is a straight line. A.3—I.14.

Now the right angle CBD is equal to the right angle FBA ; to each of these equals add the angle ABC , then the whole angle ABD is equal to the whole angle FBC .

Then in the triangles ABD , FBC , because

$$\left\{ \begin{array}{l} AB \text{ is equal to } FB, \\ BD \text{ to } BC, \\ \text{and the angle } ABD \text{ to the angle } FBC; \end{array} \right.$$

therefore the triangle ABD is equal to the triangle FBC .

C.1.—I.4.

Again, the parallelogram BL is double of the triangle ABD , for they are on the same base and between the same parallels.

F.3.—I.41.

And for a similar reason the square BG is double of the triangle FBC .

F.3.—I.41.

therefore the parallelogram BL is equal to the square BG .

Ax.6.

Similarly by joining AE , BK , it can be proved that the parallelogram CL is equal to the square CH .

But the parallelograms BL and CL together make up the square BE ,

therefore the square BE is equal to the sum of the squares BG , CH ,

that is, the square on BC is equal to the sum of the squares on BA , AC .

Q.E.D.

1. Any square is equal to half the square on its diagonal.
2. The sum of the squares on the sides of a rectangle is equal to the sum of the squares on its diagonals.
3. If the diagonals of a quadrilateral intersect at right angles the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair.
4. If the squares on two straight lines are equal, then these lines are equal.

5. ABC is an isosceles triangle, of which the angle at A is a right angle. D is the middle point of BC . Prove that the square on BC is four times the square on AD .

6. In Fig. 152 show that

- (i) F, A, K are points in the same straight line.
- (ii) B, G and C, H are points in parallel straight lines.
- (iii) A, E and B, K are points in perpendicular straight lines.

7. In Fig. 152 let EM be drawn perpendicular to KC produced. Deduce the following :

- (i) The triangles ABC, MEC are congruent.
- (ii) C is the middle point of MK .
- (iii) The sum of the squares on EK, DF is equal to five times the square on BC .

8. *The two tangents drawn from an external point to a circle are equal.*

PROBLEMS.

1. Demonstrate Fundamental Problems 22, 23 and 24.
2. To construct a square equal to the sum of two given squares.
3. To construct a square equal to the difference of two given squares.
4. To divide the given straight line AB into two parts such that the sum of the squares on them may be equal to the square on the given straight line CD . When is this impossible? (See Ex. 8, page 113.)
5. To describe on a given st. line a rectangle equal to a given triangle.
6. To construct a triangle having one of its sides equal to a given st. line, and its area equal to that of a given triangle.

HINT.--Let AB be the given st. line and CDE the given triangle. From DE , or DE produced, cut off DP equal to AB . Join PC . Through E draw EQ parallel to PC meeting DC , or DC produced, at Q . Join PQ .

CHAPTER XI.

SYMMETRY.

I.—Axial Symmetry.

Mark with ink a number of points on a sheet of paper. While the ink is still wet fold the paper over on itself so that an impression of each point may be made on its surface.

Join the corresponding points on opposite sides of the crease. Observe that the straight lines thus drawn are all bisected at right angles by the straight line which the crease represents.

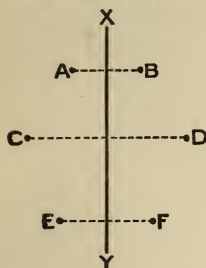


FIG. 153.

Two points are said to be **symmetrical with respect to a straight line**, called the **axis of symmetry**, when this line bisects at right angles the straight line joining the points.

Thus in Fig. 153 the pairs of points A and B, C and D, E and F are symmetrical with respect to the straight line XY as an axis.

Let ABC and ADC be two triangles whose boundaries coincide at all points. Imagine that ADC is rotated about the common side AC (produced to X and Y in the diagram) through an angle of 180° . Then the triangles ABC and ADC will lie in the same plane and on opposite sides of AC. Observe the following :

(i) In the first position the triangles coincide with each other. Hence for every point, line or angle in the one there is a corresponding point, line, or angle in the other.

(ii) The triangle ADC has undergone no alteration of shape or size by the change in its position. Hence the triangles are still congruent, having corresponding points, lines and angles as before, but on opposite sides of the straight line XY.

(iii) The straight lines joining corresponding points in the triangles are bisected at right angles by the straight line XY.

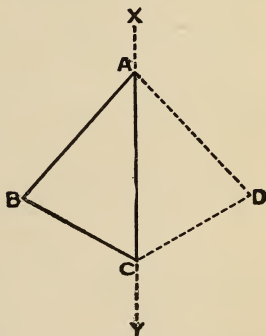


FIG. 154.

Two figures are said to be **symmetrical with respect to an axis**, when by turning one of them about that axis through an angle of 180° , it can be made to coincide with the other.

It is important to observe that when two figures are symmetrical with respect to an axis, there is for every point in the boundary of either of them a corresponding symmetrical point in the boundary of the other.

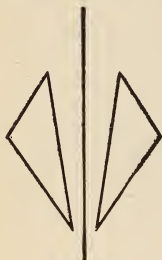


FIG. 155.

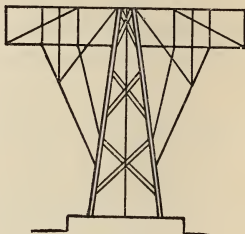


FIG. 156.

A figure is symmetrical with respect to an axis when it can be divided into two figures (parts) which are symmetrical with respect to that axis. (See Figs. 155 and 156.)

Some figures are symmetrical with respect to two or more straight lines. Thus the square ABCD in Fig. 157 has four axes of symmetry; viz., AC, BD, EF, and GH.

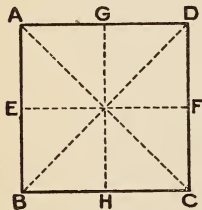


FIG. 157.

Show that an isosceles triangle is symmetrical with respect to the bisector of the vertical angle as an axis.

Draw two triangles symmetrical with respect to a given straight line XY; also, two quadrilaterals.

How many axes of symmetry has a quadrilateral made up of two isosceles triangles standing on opposite sides of a common base? An equilateral triangle? A rectangle? A square? A circle?

II.—Central Symmetry.

Let P and Q, as in Fig. 158, be any two points in a plane, and let the straight line joining them be bisected at O. Suppose OQ to be rotated in the plane through an angle of 180° about the point O; then OQ will coincide with OP the point Q falling on the point P. The points P and

Q are said to be symmetrical with respect to the point O . Similarly, if any number of other straight lines P_1Q_1 , P_2Q_2 , etc., are bisected at O , the pairs of points P_1 and Q_1 , P_2 and Q_2 , etc., are symmetrical with respect to the point O .

Two points are said to be **symmetrical with respect to a third point**, called the **centre of symmetry**, when this point bisects the straight line joining the two points.

Two figures are said to be **symmetrical with respect to a centre** when by turning one of them in the plane about that centre through an angle of 180° , it can be made to coincide with the other.

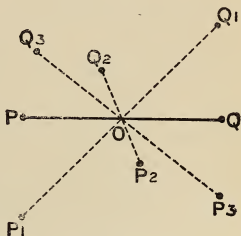


FIG. 158.

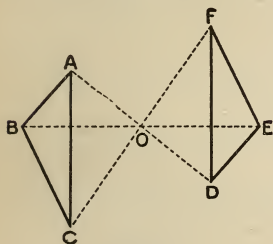


FIG. 159.

When two figures are symmetrical with respect to a centre, there is for every point in the boundary of either of them a corresponding symmetrical point in the boundary of the other. Thus in Fig. 159 the triangles ABC and DEF are symmetrical with respect to the centre O .

Show that by rotating one of these triangles about the centre of symmetry it can be made to coincide with the other.

When a single figure is symmetrical with respect to a centre, every straight line drawn through that centre cuts the figure in two points which are symmetrical with respect to the centre.

Thus the rectangle $ABCD$, as in Fig. 160, is symmetrical with respect to the centre O .

Draw four figures symmetrical with respect to a centre.

Show that the points in which any straight line through the centre cuts corresponding lines are corresponding points.

Draw two figures each having two axes of symmetry at right angles to each other. Show that each has a centre of symmetry.

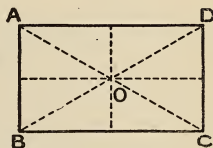


FIG. 160.

ANALYSIS AND SYNTHESIS.*

I.—Theorems.

In demonstrating the truth of a theorem we usually take the hypothesis, or what is given, as our starting-point in reasoning, and we proceed from it step by step by the aid of known truths until we arrive at the conclusion. When we reason in this way from accepted truths to new truths, the process is called *Synthesis*.

It frequently happens, however, that we find difficulty in founding on accepted truths an argument leading directly from the hypothesis to the conclusion. When this is the case we reverse the order stated, by assuming that the conclusion is true and reasoning from it to the hypothesis. This process is called *Analysis*.

If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

Let ABC be a triangle of which the angle BAC is equal to the sum of the angles ABC , ACB .

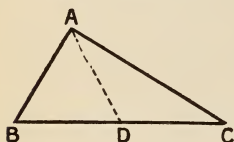


FIG. 161.

It is required to prove that the triangle ABC can be divided into two isosceles triangles.

Analysis.—Assuming that what is to be proved is true, let the triangle ABC be divided by the straight line AD into the two isosceles triangles DAB , DAC having the common vertex D in BC .

*The word *Analysis* literally means “a breaking-up”; the word *Synthesis*, “a putting-together”. When we wish to understand the construction of a piece of mechanism, as a bicycle or a clock, we first look at it carefully as a whole, and then we proceed to take it apart in order that each part may be examined by itself. After studying the separate parts we put them together again so that we may see how each fits in with the others in making up the whole.

When we simplify a whole by taking it apart and fix our attention on the parts separately, the process is termed *analysis*. On the other hand, when we put together the parts which constitute a whole, fixing our attention on the relation which each bears to that whole, the process is termed *synthesis*.

Then in the triangle DAB, because DB is equal to DA, *Assumed.*
 therefore the angle DAB is equal to the angle DBA, *C.4.—I.5.*
 that is, the angle DAB is equal to the angle ABC.

Similarly it may be shown that the angle DAC is equal to the angle ACB.

But the whole angle BAC is equal to the sum of the angles DAB, DAC; *Ax.9.*

Therefore the angle BAC is equal to the sum of the angles ABC, ACB. *Ax.1.*

We have thus shown that the hypothesis is consistent with the conclusion, and we have also seen how to effect the necessary construction. To complete the demonstration we must retrace the steps just taken, thus showing that the conclusion can be deduced from the hypothesis.

Synthesis.—At the point A make the angle BAD equal to the angle ABC, and let AD meet BC at D.

Because the whole angle BAC is divided into the two angles DAB, DAC, therefore the sum of the angles DAB, DAC is equal to the angle BAC. *Ax.9.*

Again, the angle BAC is equal to the sum of the angles ABC, ACB, *Given.*
 therefore the sum of the angles DAB, DAC is equal to the sum of the angles ABC, ACB. *Ax.1.*

But the angle DAB is equal to the angle ABC; *Const.*
 therefore, taking equals from equals, the remaining angle DAC is equal to the remaining angle ACB. *Ax.3.*

Then in the triangle DAB, because the angles DAB, DBA are equal, *Const.*
 therefore the sides DB, DA are equal; *C.6.—I.6.*
 that is, the triangle is isosceles. *Def.49.*

Also, in the triangle DAC, because the angles DAC, DCA are equal, *Proved.*
 therefore the sides DC, DA are equal; *C.6.—I.6.*
 that is, the triangle is isosceles. *Def.49.*

The conclusion is thus deduced from the hypothesis, which establishes the theorem.

In analyzing a theorem we attempt to answer this question: 'If the theorem is true what consequences does it lead to?' We suppose that the theorem is true, and we proceed by the

aid of established truths to determine what the consequences of this supposition are. Several different results may be thus obtained: three of these we shall consider.

(i) If we deduce a conclusion which contradicts a known truth, then we are certain that the theorem is not a true one; for we have arrived at a *reductio ad absurdum*. (See pp. 71-72.)

(ii) If we reach a conclusion that is known to be true, then we try either to deduce it from the hypothesis, or to reason from it back to the hypothesis.

(iii) If we deduce the hypothesis, as in the foregoing illustration, we then reverse, if possible, the order of steps in the analysis, and thus prove the theorem by synthesis.*

NOTE.—If the steps are not reversible no conclusion can be drawn.

Generally speaking, the simplest and best way to analyze a theorem is to substitute for it another upon which it depends, and so on until a known truth is arrived at.

If two exterior angles of a triangle formed by producing a side both ways are equal, the triangle is isosceles.

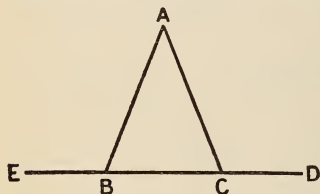


FIG. 162.

Let ABC be a triangle of which the side BC is produced both ways to D and E, making the exterior angles ACD, ABE equal.

It is required to prove that the triangle ABC is isosceles.

(i) The theorem is true, if AB is equal to AC. Def. 49.

(ii) AB is equal to AC, if the angle ACB is equal to the angle ABC. C.6—I.6.

(iii) The angle ACB is equal to the angle ABC, if the exterior angle ACD is equal to the exterior angle ABE. A.1.—Cor.1.

(iv) But the exterior angle ACD is equal to the exterior angle ABE. Given.

(v) Therefore the theorem is true.

Explain the argument fully.

* For a discussion of "The Ancient Geometrical Analysis," see Pott's *Euclid*, large edition, pp. 288-293, or school edition, pp. 64-68.

Stated generally, the argument is as follows:—*P is Q* is true, if *R is S* is true. *R is S* is true, if *T is U* is true. But *T is U* is known to be true. Therefore *R is S*, and *P is Q* are true.

II.—Problems.

In solving a problem it is best to begin by making a careful study of the properties of the construction to be effected. We assume that the diagram is drawn as required, and we proceed to examine it, noting particularly the relations which exist between what is known and what is unknown. The purpose of this process of analysis is to find out how the required construction can be made.

Having determined by analysis what known constructions can be used in solving the problem, we then proceed to apply these in accordance with the given conditions, until the required construction is effected. The different steps in this synthetic (putting-together) process make up what is known as the *construction* of the problem. (See page 77.)

The *demonstration* consists in showing by the aid of known truths that the construction conforms to the requirements of the problem. Following the demonstration there should be a discussion of particular cases; for in many instances the number of possible solutions, or even the possibility of obtaining a solution, depends upon the values of the given magnitudes, the positions of the given points, etc.

(i) *Through a given point to draw a straight line parallel to a given straight line.*

Let *AB* be the given straight line, and *P* the given point. It is required to draw through *P* a straight line parallel to *AB*.

Analysis.—Suppose the straight line *CD* to be drawn through *P* parallel to *AB*.

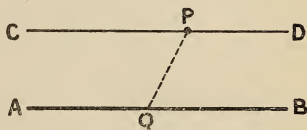


FIG. 163.

Now a straight line drawn from *P* to any point *Q* in *AB* will make the angle *CPQ* equal to the alternate angle *BQP*, *D.2.—I.29,*

Hence the direction of the straight line required is determined by joining P to any point Q in AB , and making the angle CPQ equal to the angle BQP .

Synthesis.—In AB take any point Q , and join PQ . At the point P in PQ make the angle CPQ equal to the angle BQP , and alternate to it. Produce CP to D .

Then CD is parallel to AB .

Because the straight line PQ , meeting the straight lines CD , AB , makes the alternate angles CPQ , BQP equal; *Const.*

therefore CD is parallel to AB .

D.2.—I.29.

(ii) Let ABC be a given triangle: it is required to find in AB a point P , and in AC a point Q , such that PQ , AP , CQ may be equal to one another.

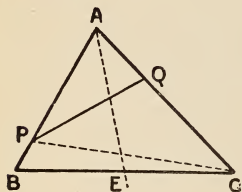


FIG. 164.

Analysis.—Suppose PQ to be drawn as required. Join CP .

Then in the triangle PAQ because PA is equal to PQ , *Assumed.*

therefore the angle PQA is equal to the angle PAQ . *C.4.—I.5.*

Similarly it may be shown that in the triangle QPC the angle QPC is equal to the angle QCP .

But the exterior angle PQA is equal to the sum of the angles QPC , QCP ; *D.3.—I.32.*

therefore the angle PQA is double of the angle QCP .

We thus see that the positions of the points P and Q may be found by making the angle ACP equal to half the known angle BAC , and the angle CPQ equal to the angle ACP .

Synthesis.—Bisect the angle BAC by the straight line AE , meeting BC at E . Make the angle ACP equal to the angle CAE , CP meeting AB at P . Make the angle CPQ equal to the angle ACP , PQ meeting AC at Q .

Then P and Q are the points required.

Write the proof in full.

Frequently the solution of a problem consists in locating points which lie in lines which are known, or easily deter-

mined. In such cases the method of analysis can be used with much advantage. (See pp. 81-82.)

To find a point which shall be equidistant from two given points and at a given distance from another given point.

Let P , Q and R be the given points and d the given distance.

It is required to find a point X which shall be equidistant from P and Q , and at the distance d from R .

Analysis.—Assume that the point X satisfies the given conditions.

Because X is equidistant from P and Q it must lie in the perpendicular bisector of PQ .

Again because X is at the given distance d from R it must lie in the circumference of the circle whose centre is R and whose radius is equal to d .

Hence the point X must be one of the points of intersection of the loci thus determined.

Synthesis.—Find the locus of a point equidistant from the given points P and Q . (See Locus 2, p. 82.)

Find the locus of a point at the given distance d from the given point R . (See Locus 1, p. 82.)

Give the proof. Under what condition does the problem admit of two solutions? Of only one solution? Of no solution?

The following general directions may serve as a guide in solving problems:

(a) Read the problem carefully so that its meaning may be fully comprehended.

(b) Make a diagram corresponding to the given conditions, and then complete it as if the problem were solved.

(c) Keeping in mind what is known about the diagram, study it carefully and note the relations of lines, angles, etc. The object of this is to find out, if possible, upon what known theorems or problems the assumed solution depends.

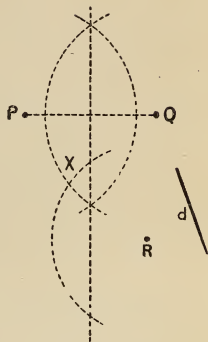


FIG. 165.

(*d*) In case no relation leading to the solution of the problem is discovered, add to the construction by drawing such lines, circles, etc., as may be required, and then continue the investigation as before.

(*e*) When the steps in solving the problem are determined, make the construction by means of postulates, or problems already solved.

(*f*) Prove that the construction conforms to the requirements of the problem.

EXERCISES. REVIEW.

1. The bisectors of vertically opposite angles are in the same straight line.

2. From a given point only one perpendicular to a given straight line can be drawn.

3. What is meant by the *locus* of a point? Find the locus of a point at a given distance from a given point.

4. In a circle place a chord equal to a given straight line.

5. *The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and of all others that which is nearer to the perpendicular is less than that which is more remote.*

6. (i) *Every point equidistant from the ends of a given straight line is in the perpendicular bisector of the given straight line.*

(ii) *Every point not in the perpendicular bisector of a given straight line is unequally distant from the ends of the given straight line.*

7. Find the locus of a point equidistant from two given points.

8. ABC , DBC are two isosceles triangles on opposite sides of the common base BC . O is the middle point of BC . Show that the points A , O , and D are in the same straight line.

[When any number of points lie in the same straight line they are said to be **collinear**.]

9. Find the locus of the vertices of isosceles triangles on a common base.

10. On the same base and on the same side of it there cannot be more than one equilateral triangle,

11. In a given straight line find a point equidistant from two given points.

12. The perpendiculars to any two sides of a triangle intersect each other.

13. *The perpendiculars to the three sides of a triangle drawn through the middle points of the sides, meet at a point.*

[The point at which the perpendicular bisectors of the three sides of a triangle meet is called the **circum-centre** of the triangle.]

14. Find a point equidistant from three given points.

15. A straight line cannot cut the circumference of a circle at more than two points.

16. The straight line joining the centre of a circle to the middle point of any chord is perpendicular to that chord.

17. *The centre of a circle lies in the perpendicular bisector of any chord.*

18. Find the centre of a circle whose circumference shall pass through two given points and whose radius shall be equal to a given straight line.

19. Describe a circle about a given triangle.

20. Describe a circle whose circumference shall pass through two given points and whose centre shall be in a given straight line.

21. The straight line drawn perpendicular to a diameter of a circle at either of its extremities is a tangent to the circle.

22. Only one tangent to a circle can be drawn through a point in the circumference.

23. The straight line drawn from the vertex of a triangle to the middle point of the base is less than half the sum of the other two sides.

24. Any straight line drawn from the vertex of a triangle to a point in the base is less than the greater of the two sides, or than either if they are equal.

Hence show that a chord of a circle lies within the circle.

25. The sum of the three medians of a triangle is less than the perimeter of the triangle.

26. *The difference of any two sides of a triangle is less than the third side.*

27. ABC , DBC are two triangles on the same base BC and on the same side of it. If AB , DB are equal, then AC , DC are unequal.

28. From two given points on the same side of a given straight line draw two straight lines meeting in the given straight line and making equal angles with it.

ANALYSIS.—Let P and Q be the given points and AB the given straight line. Suppose X to be a point in AB such that the angles AXP , BXQ are equal. Produce QX to P_1 making XP_1 equal to XP . Join PP_1 . Show that P and P_1 are symmetrical with respect to AB , etc.

29. In a given straight line find a point such that the sum of its distances from two given points on the same side of the given straight line may be the least possible.

HINT.—Make the construction as in the preceding problem. Show that if any other point Y be taken in AB , the sum of PY , QY is greater than the sum of PX , QX . Apply *C.S.—I.20*.

30. The perpendicular drawn from the centre of a circle on any chord bisects that chord.

31. Through a given point within a circle, which is not the centre, draw a chord so that it shall be bisected at the given point.

32. Let P , Q , and R be three given points. Through P draw a straight line so that the part of it intercepted by the perpendiculars drawn on it from Q and R may be bisected at P .

HINT.—Find a point Q_1 such that Q and Q_1 may be symmetrical with respect to the point P . Join Q_1R , etc.

33. Show how *C.10.—I.26* may be applied practically in finding the width of a river without crossing it.

34. (i) *Every point in the bisector of an angle is equidistant from the arms of the angle.*

(ii) *Every point not in the bisector of an angle is unequally distant from the arms of the angle.*

35. Find a point equidistant from two given points and also equidistant from two intersecting straight lines.

36. In a given straight line find a point equidistant from two given intersecting straight lines. Is a solution always possible?

37. *The bisectors of the three angles of a triangle are concurrent.*

[The point at which bisectors of the three angles of a triangle meet is called the **in-centre** of the triangle.]

38. Show that the bisectors of two exterior angles of a triangle must intersect each other.

39. From the point at which the bisectors of two exterior angles of a triangle intersect perpendiculars are drawn to the sides, or the sides produced. Show that these perpendiculars are equal.

40. *The bisectors of two exterior angles of a triangle, and the bisector of the interior opposite angle, are concurrent.*

[The point at which the bisectors of two exterior angles of a triangle, and the bisector of the interior opposite angle meet, is called an **ex-centre** of the triangle.]

41. Find a point equidistant from the three sides of a given triangle.

42. Show that a right-angled triangle can be divided into two isosceles triangles.

43. *In a right-angled triangle the straight line joining the right angle to the middle point of the hypotenuse is equal to half the hypotenuse.*

44. In the triangle **ABC**, **AD** and **BE** are perpendiculars drawn from **A** and **B** to the opposite sides respectively. Show that the points **D** and **E** are equidistant from the middle point of **AB**.

45. If one of the acute angles of a right-angled triangle is double of the other, the hypotenuse is double of the shortest side.

46. *Each angle of an equilateral triangle is equal to two-thirds of a right angle.*

47. Trisect a right angle.

48. The sum of the angles of a quadrilateral is equal to four right angles.

49. The sum of the exterior angles of a quadrilateral is equal to four right angles.

50. *The perpendiculars drawn from points in one of two parallel straight lines to the other are equal.*

51. Find the locus of a point at a given distance from a given straight line.

52. What must be the path of a moving point so that it will always be equidistant from two given points **A** and **B**? Always nearer to **A** than to **B**? Equidistant from **A** and **B** but once?

53. Find a point equidistant from two given points and at a given distance from a given straight line. How many solutions are possible? In what case is a solution impossible?

54. Two angles whose arms are parallel, respectively, are either equal or supplementary.

55. Two angles whose arms are perpendicular, respectively, are either equal or supplementary.

56. If a straight line is perpendicular to any one of a series of parallel straight lines, it is perpendicular to every one of the series.

57. *If any number of parallel straight lines intercept equal parts of one transversal they intercept equal parts of every other transversal.*

58. Trisect a given straight line.

59. *The diagonals of a parallelogram bisect each other.*

60. *The straight line joining the middle points of two sides of a triangle is parallel to the third side, and equal to half of it.*

61. A straight line drawn parallel to the base of a triangle through the middle point of one of the sides bisects the remaining side.

62. The perpendiculars drawn from the middle points of two sides of a triangle on the third side are equal.

63. The straight lines joining the middle points of the sides of an isosceles triangle form an isosceles triangle.

64. The straight lines joining the middle points of any triangle form a triangle equiangular to the whole triangle.

65. To construct a triangle when the middle points of its sides are given.

66. To construct a triangle equal to one-sixteenth of a given triangle and equiangular to it.

67. The middle points of the sides of a quadrilateral are the vertices of a parallelogram whose area is half that of the quadrilateral.

68. The middle points of the sides of a rhombus are the vertices of a rectangle.

69. *The three medians of a triangle are concurrent at a point of trisection of each median.*

[The point at which the medians of a triangle meet is called the **centroid** of the triangle.]

70. ABCD is a parallelogram. E and F are the middle points of the opposite sides AD, BC, respectively. BE and DF cut the diagonal AC at G and H. Show that AC is trisected at these points.

71. *The perpendiculars from the vertices of a triangle to the opposite sides are concurrent.*

HINT.—Through the vertices of the triangle draw straight lines parallel to the opposite sides, respectively. Apply Nos. 43 and 13.

[The point at which the perpendiculars from the vertices of a triangle to the opposite sides meet is called the **orthocentre** of the triangle.]

72. *Every straight line drawn through the middle point of either diagonal of a parallelogram bisects the parallelogram.*

73. Through a given point draw a straight line bisecting a given parallelogram.

74. If a perpendicular is drawn from the vertex of a triangle to the base, the difference of the squares on the segments of the base is equal to the difference of the squares on the two sides.

75. A series of right-angled triangles have a common base. Show that their vertices lie in the circumference of a circle.

PART II.

EUCLID, BOOK I.

EUCLID.

Euclid, a famous mathematician of the first Alexandrian school, lived in the third century B.C. Very few facts regarding his life have been recorded; of the time and place of his birth, his parentage, or his early education nothing is definitely known. It is generally supposed, however, that he was of Greek descent, and that he received his early training at Athens.

Euclid is said to have founded the first mathematical school of Alexandria about 300 B.C. Some years after this he undertook to collect and arrange in systematic order the scattered principles and truths of elementary Geometry, in so far as they were known at the time. He thus produced that most remarkable work known as the *Elements*.

Although Euclid made free use of the discoveries of his predecessors in the preparation of the *Elements*, the work cannot be regarded as a mere compilation, for he not only added many new demonstrations of his own, but he also completed, simplified, and re-stated many of the demonstrations of previous writers. His marvellous power of analysis enabled him to discriminate between axiomatic and derivable truths, and to trace the latter back to the fundamental principles underlying them. Taking simple, self-evident truths as the starting-point, and reasoning from them to other truths, and so on, he built up a system of Geometry destined to stand the test of time.

INTRODUCTION.

Geometry is the branch of mathematics that deals with *size*, *shape* and *position*. The only things we can conceive of as possessing these characteristics and no others, are space magnitudes—solids, surfaces, lines and angles. These, together with points (a point denotes position only), constitute the subject-matter of pure Geometry.

Plane Geometry treats of such magnitudes as may be described on a plane surface.

I.—The Definitions.

We arrange things into classes according to their qualities. Those which have some quality or set of qualities in common constitute a class, and they are known by a common or class name. Tree, book, circle, are examples of class names.

In order that a name may convey to us a definite meaning we must know the essential characteristics of the thing or class of things for which it stands. One reason why we so often fail to think clearly ourselves, or grasp the thoughts expressed by others, is that the meaning of the language used is not always perfectly clear to us. Now, as geometry is an exact science, it is necessary that the exact meaning of every term employed should be known. The purpose of the definitions of geometry is to tell us exactly what things, and classes of things, the subject deals with.

Jevons says: "*By a definition we mean a precise statement of the qualities which are just sufficient to mark out a class, and to tell us exactly what things belong to a class and what do not,*"

Note the words "just sufficient." A definition does not explain the properties of the thing defined ; it merely gives such marks as enable us to distinguish that thing from all other things not belonging to the same class. Suppose, for example, we were to define a parallelogram to be a quadrilateral whose *opposite sides are equal and parallel*, we would violate the principle here stated ; for we would thus include a property of the parallelogram which follows from the fact that its opposite sides are *parallel*. Although it is quite true that the opposite sides of a parallelogram are *equal*, a definition including this property could not be admitted into any logical argument. (See Def. 55.)

[As the definitions used in Euclid, Book I, are arranged in Part I so that they may be easily referred to, it is unnecessary to repeat them here.]

II.—The Postulates.

The postulates are statements of simple constructions which are assumed as possible. They may be regarded as the first principles of geometrical drawing.

I. A straight line may be drawn from any one point to any other point.

It is assumed that a straight line may exist anywhere in space. This implies that space is continuous or everywhere the same. In geometrical representation this postulate permits the use of the ungraduated ruler or straight-edge as a guiding instrument in drawing a straight line between any two given points.

II. A terminated straight line may be produced to any length in a straight line.

It is here assumed that a straight line may be of unlimited length. This implies that space is not only continuous but also unlimited, otherwise the existence of an unlimited line in it would be inconceivable. Again, as the terminated straight line may lie between any two points whatsoever, the unlimited extension of this line implies that space is extended without limit in all directions. Practically, this postulate gives

permission to use the straight-edge as a guiding instrument in extending the length of any given straight line in either direction, or in both directions.

III. A circle may be described from any centre at any distance from that centre.

In this postulate the existence of an unlimited plane passing through a point is assumed, otherwise it would be impossible to describe a circle having a radius of any given length howsoever great.

This postulate gives permission to use compasses in describing a circle having any given point as its centre and its radius equal to a given straight line already drawn from that point. Euclid does not permit the use of compasses for transferring distances.

It must be remembered that the figures of pure geometry exist only in the mind ; we *represent* these by means of diagrams in order to make it easier for us to think about them.

III.—The Axioms.

The axioms are the fundamental truths or principles upon which the science of geometry rests. As one of the objects of geometry is to discover first principles, no statement can be accepted as an axiom merely because it is a self-evident truth : it must be so simple as not to admit of proof. The requirements of an axiom are stated on page 42.

1. Things which are equal to the same thing, or to equal things, are equal to one another.

2. If equals be added to equals, the sums are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the sums are unequal in the same sense.

5. If equals be taken from unequals, the remainders are unequal in the same sense.

6. Things which are doubles of the same thing, or of equal things, are equal to one another,

7. Things which are halves of the same thing, or of equal things, are equal to one another.

A. Magnitudes can be moved freely in space, without alteration of shape, or size.

[This truth is assumed by Euclid though he does not state it as an axiom.]

As we are already aware, a geometrical magnitude is such that it cannot be seen, felt, handled, etc. It follows that such a magnitude cannot be actually moved from one position to another; hence the motion implied in the axiom can only be *thought of*.

8. Magnitudes that can be made to coincide with one another, are equal.

The converse of this axiom as applied to straight lines and angles is assumed by Euclid.

(i) Equal straight lines can be made to coincide with one another.

(ii) Equal angles can be made to coincide with one another.

Both of these are used in the proof of Euc. I. 4.

9. The whole is greater than its part, and equal to all its parts, taken together.*

10. Two straight lines cannot enclose a space.

Euclid's eleventh axiom is as follows: "*All right angles are equal to one another.*" As this statement admits of proof it cannot be regarded as an axiom. It may be proved either by direct superposition, or it may be deduced from the theorem, "*All straight angles are equal,*" as shown on page 68. (As Euclid does not recognize the existence of a straight angle, the theorem is proved independently, on page 192.)

11. Through a point there cannot be more than one straight line parallel to a given straight line.

Axiom 11 is substituted for Euclid's twelfth axiom, which is as follows: "*If a straight line meet two straight lines so as to make the interior angles on one side of it together less than two right angles, these straight lines will meet if continually produced on the side on which are the angles which are together less than two right angles.*"

* "Some of the axioms come so near to definitions in their nature that their place may be considered as doubtful. Such are, 'the whole is greater than its part,' and 'magnitudes which entirely coincide are equal to one another,'"

De Morgan; The Study of Mathematics, page 208.

Objection has been taken to Euclid's axiom for reasons which have been often stated :

- (i) It cannot be regarded as self-evident.
- (ii) It can be deduced from simpler truths.*

Another objection has been urged against this axiom, viz., that it is the converse of a theorem which requires demonstration ; but the same may be said of every axiom proposed as a substitute for it.

B. A line drawn between two points, one being within, and the other without a closed figure, cuts the boundary of the figure.

[This truth is assumed by Euclid, though not stated as an axiom.]

From Axiom B deduce the following :

- (i) A straight line of unlimited length drawn through a point within a closed figure cuts the boundary of the figure at two points at least.
- (ii) If there are any two points in the boundary of a closed figure, such that one lies within and the other without another closed figure, the boundaries of these figures intersect at two points at least.

The first seven axioms, and also the ninth, are applicable to all magnitudes. They are, therefore, called *general axioms*.

Axioms A, 8, 10, 11 and B, are called *geometrical axioms* because they can be applied to geometrical magnitudes only.

IV.—Inductive and Deductive Reasoning.

A boy throws a piece of wood into a pond. The wood floats in the water. He does the same thing with different kinds of wood, and he finds that in each case the wood floats. He comes to the conclusion that all kinds of wood float in water. He has performed a simple kind of reasoning based on the observations and experiments he has made. But he cannot say with any certainty that the conclusion is true, for he has not tested all the different kinds of wood. A great deal of the knowledge we have has been gained in much the

* It is not contended that the axiom here given, as a substitute for Euclid's, removes entirely the foregoing objections. This much may be said in its favor, however,—it is easily comprehended and easily applied.

same way. We have learned that "dogs bark," "trees have roots," "stones are heavy," and many other things by observing particular dogs, trees, stones, etc., and drawing general conclusions from our observations.

Again, suppose we draw two straight lines AB , CD cutting each other at the point E , as in Fig. 65, page 70, and measure the vertically opposite angles AEC , BED with a protractor, we shall find that these angles are equal. We then draw other straight lines cutting each other and we find that in each case the same thing holds good. We then come to the conclusion that in all cases the vertically opposite angles formed by two straight lines cutting each other are equal. Observe that this conclusion, like the former, is based on the examination of particular cases, and that while we may be fully convinced of the truth of it, we cannot say we have proved it to be true in all possible cases.

When we come to a conclusion about a whole class of things as the result of seeing, hearing, handling, measuring, etc., particular things belonging to that class, the process of reasoning is said to be *inductive*. This method of procedure is very useful in the study of geometry, for it enables us to find out many facts for ourselves in a simple way; but we must bear in mind that it does not enable us to prove any fact to be *true*. As we shall see presently, the truths of geometry are established by a method entirely different from this one.*

* Although the origin of geometry is hidden in the past, yet we have every reason to believe that in its earliest stages many truths were discovered in a practical way, that is, by marking out and measuring geometrical figures—lines, angles, triangles, etc. The earliest definite information on the subject is furnished by the Rhind papyrus (belonging to the Rhind collection in the British Museum), which was written by an Egyptian priest at some time between 1700 B.C. and 1100 B.C. This document indicates clearly that geometry originally consisted of a system of rules derived from experiments in practical measurement. Ball, in his *Short History of Mathematics*, page 8, remarks thus: "It is noticeable that all the specimens of Egyptian geometry which we possess deal only with particular numerical problems and not with general theorems; and even if a result be stated as universally true, it was probably proved to be so only by a wide induction." According to Herodotus, Sesostris an Egyptian monarch, who lived during

Consider the following statements :

(a) All Canadians are British subjects.

(b) Jones is a Canadian.

What conclusion can we draw from these two statements? Evidently it is :

(c) Jones is a British subject.

We have thus deduced from the two statements (a) and (b) a third statement (c) which is different from (a) and (b); and we know that if (a) and (b) are both true, (c) must also be true.

This indicates the way in which we proceed in proving the truths of geometry. Beginning with known truths or principles, we reason from these to new truths. Such a process of reasoning is said to be *deductive*.

A very important question arises here. According to this method of procedure we must have known truths to reason from before we can take a single step in advance. How are we to get a starting-point? We overcome this difficulty by admitting without proof that certain simple, self-evident statements are true. These self-evident truths form the foundation upon which the science of geometry rests.†

the 14th century B.C., divided up the country amongst the people, allotting to each person a definite portion of land upon which a tax was levied. But, owing to the overflow of the Nile each year, many of the landmarks were removed, and in some cases the allotments themselves, or parts of them, were covered with water; hence an annual inspection of the survey became necessary. This gave an impetus to the systematic study of geometry in Egypt. The derivation of the word geometry—from two Greek words, $\gamma\epsilon$, the earth, and $\mu\acute{\epsilon}\tau\rho\nu$, a measure—points to the conclusion that geometry in its experimental stage was developed chiefly in connection with the measurement of land.

†While much credit is due to the Egyptians for the advancement they made in geometry on its practical side, the glory, for such it is, of discovering and stating the principles upon which it rests belongs to the Greeks. Not only did they state the universal truths which lie at the foundation of the subject, but they deduced from these many other truths and arranged the whole in logical order, thus giving to geometry the status of an exact science. Ball in the excellent work already referred to, says: "Now the Greek geometers, as far as we can judge by their extant works, always dealt with the science as an abstract one: they sought for theorems which should be absolutely true."

V.—The Syllogism.

When a process of reasoning is expressed in the form logic requires, it is called an **argument** or **syllogism**, thus :

All Canadians are British subjects.

Jones is a Canadian.

Therefore, Jones is a British subject.

We shall now examine this syllogism.

It consists of three statements such that if the first and second are true, the third is necessarily true. Each of these statements is called a **proposition**.

The first two propositions are called the *premises*, and the third the *conclusion*.

As a proposition expresses a thought, it consists of three distinct parts—the *subject*, or thing spoken of ; the *predicate*, or thing asserted of the subject, and the *copu'a*, which shows the connection between the subject and predicate. Thus, in the proposition “coal is black,” *coal* is the subject, *black* the predicate, and *is* the copula. The subject and predicate of a proposition are frequently called *terms*.

Look at the first proposition. It tells us that every Canadian is included among British subjects. When a proposition thus asserts that the whole class of things denoted by the subject is included in the class of things denoted by the predicate, the proposition is said to be *universal*. Give other illustrations of universal propositions.

The second proposition asserts that Jones is one of the individuals belonging to the class denoted by the term Canadians. When the subject of a proposition denotes a single individual, the proposition is said to be *singular*. Give other illustrations of singular propositions.

As the syllogism consists of three propositions it contains six terms, but, as each term occurs twice, the syllogism con-

tains only *three* different terms, viz.: "British subjects," "Canadians" and "Jones." The predicate of the conclusion is called the *major term* and the subject of the conclusion the *minor term*. The term which does not appear in the conclusion is called the *middle term*.

We are now in a position to consider how the conclusion is reached. The first premise tells us that all Canadians (every individual) are included in the larger class, known as British subjects. The second premise tells us that Jones is one of the individuals in the class known as Canadians. Combining in thought these two premises, we arrive at the conclusion that Jones is one of the individuals included among British subjects. This process of reasoning may be illustrated by circles, as in the diagram.

The premise containing the predicate of the conclusion (the major term) is called the *major premise*. It is the general statement upon which the argument is based. The premise, containing the subject of the conclusion (the minor term), is called the *minor premise*. It brings the subject of the conclusion within the scope of the major premise; in other words, the minor premise shows that what is asserted of the subject of the major premise may also be asserted of the subject of the conclusion. The whole object of the syllogism is to compare the minor term with the major term by considering the relation which each bears to the middle term.

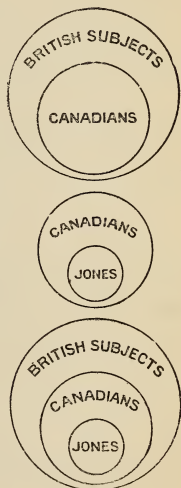


Fig. 166.

VI.—Geometrical Reasoning.

The syllogism which we have just considered may be regarded as a type of geometrical reasoning. The only distinc-

tion between a geometrical syllogism and any other arises out of the nature of the facts which form the subject-matter of geometry. We shall now consider what the facts reasoned about in geometry are, and how they are obtained.

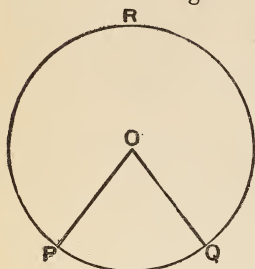


FIG. 167.

Let PQR be a circle of which Q is the centre. Join OP and OQ. It is required to show that the radii OP and OQ are equal.

In order to show the complete process of reasoning we must express it in the form of a syllogism.

SYLLOGISM A.

All radii of a circle are equal. (Major premise.)

OP and OQ are radii of a circle. (Minor premise.)

\therefore OP and OQ are equal. (Conclusion.)

Observe that the major premise of Syllogism A is *Def. 33*, and the minor premise, the *hypothesis*, or *given condition*.

Let AB and CD be two given straight lines, each of which is double of the given straight line EF. Prove that AB is equal to CD.

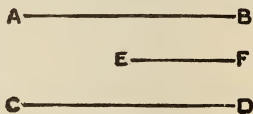


FIG. 168.

SYLLOGISM B.

Things which are doubles of the same thing are equal to one another. (Major premise.)

AB and CD are each double of EF. (Minor premise.)

\therefore AB and CD are equal. (Conclusion.)

The major premise of Syllogism B is *Axiom 6*. The minor premise is the *hypothesis*.

Let ABC be a triangle, of which the side BC is produced to D . It is required to prove that the sum of the angles ACD , ACB , is greater than the sum of the angles ABC , ACB .

SYLLOGISM C.

If a side of a triangle is produced, then the exterior angle is greater than either of the interior opposite angles.

(Major premise.)

The angle ACD is an exterior angle of the triangle ABC , and the angle ABC is one of the interior opposite angles.

\therefore the angle ACD is greater than the angle ABC . (Conclusion.)

SYLLOGISM D.

If equals be added to unequals the sums are unequal in the same sense. (Major premise.)

The angle ACB is added to each of the unequal angles ACD , ABC . (Minor premise.)

\therefore the sum of the angles ACD , ACB , is greater than the sum of the angles ABC , ACB . (Conclusion.)

The major premise of Syllogism C is a *truth which has already been demonstrated*: the minor premise is the *given condition*. The major premise of Syllogism D is an *axiom*, and the minor premise is *based on the conclusion of Syllogism C*. We thus see that the conclusion of a syllogism may become a premise of a succeeding one, and so on. In this way a chain of connected syllogisms may be formed.

Summing up the foregoing results, we find that the facts upon which geometrical reasoning is based are as follows:

1. *Definitions*. We thus see the importance of constructing all definitions in accordance with the requirements of correct thinking. Evidently no definition should be admitted until it is certain that the thing defined is really possible.

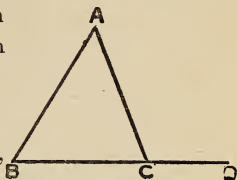


FIG. 169.

2. *Conditions which are given or assumed.* Every proposition states (in effect at least) that if certain conditions are given, certain consequences must follow. Such given conditions are taken as a basis of reasoning in proving that the theorem is true. The same may be said regarding conditions assumed but not stated in the theorem, as in cases of indirect demonstrations. (See page 71.)

In order to prove a theorem it is frequently necessary to make new conditions by drawing lines, etc. Such conditions are introduced into the argument merely for purposes of demonstration. (See foot-note, page 69.)

3. *Axioms.* As the axioms are the fundamental principles upon which geometrical reasoning is based, it is clear that they must be not only self-evident, but also incapable of proof.

4. *Truths already established.* From the axioms as first principles new truths are deduced, and from these still other truths, and so on. In this way the science of geometry has been built up. As each truth is deduced from truths already admitted or proved, it is beyond question; hence it may be used subsequently as a basis of reasoning in establishing other truths.

Although every process of reasoning by which a truth of geometry is established can be reduced to the form of a syllogism, it is not usually so expressed. For example, when we say "because AB, AC are radii of the same circle, therefore AB is equal to AC," the major premise, viz., "all radii of a circle are equal," is understood. Whenever there is doubt as to the validity of a conclusion, the premises from which it is deduced should be clearly stated.*

* While it is unnecessary to write out many demonstrations in syllogistic form, yet there is no more valuable exercise for the student than to review a demonstration now and then, supplying the premises which have been suppressed. The student's ability to do this furnishes the very best evidence that he has fully mastered the demonstration.

VII.—Propositions.

Euclid's *Elements** is divided into thirteen books, the first four and the sixth of which deal with plane geometry. Each book is divided into parts called **propositions**.

The propositions are of two kinds, called **problems** and **theorems**.

A **theorem** is a statement of a geometrical truth which is to be demonstrated. (See page 65.)

A **problem** is a statement of a geometrical construction to be effected. (See page 77.)

A proposition consists of the following parts :

(i) **General enunciation.** This is a statement in general terms of the given conditions, and of what has to be done or proved.

(ii) **Particular enunciation.** The particular enunciation repeats what is stated in the general enunciation, referring this to a diagram which is drawn.

(iii) **Construction.** The construction gives directions for drawing such lines or circles as may be necessary in order to solve the problem or demonstrate the truth of the theorem, as the case may be.

[The learner will observe that Euclid does not make use of constructions other than those permitted by the postulates, until he has shown how to effect them.]

(iv) **Demonstration or proof.** The demonstration is the process of reasoning by which it is shown that the construction effected conforms to the requirements of the problem, or that the conclusion of the theorem follows as a necessary consequence from the hypothesis. (See pp. 71–72.)

* Euclid's *Elements* comprises thirteen books, of which the first four and the sixth treat of plane geometry ; the fifth, of the theory of proportion ; the seventh, eighth and ninth, of the theory of numbers ; the tenth, of incommensurable magnitudes ; the eleventh and twelfth, of the principles of solid geometry ; and the thirteenth of incommensurable lines in plane figures, and of the regular solids.

SYMBOLS AND ABBREVIATIONS.

\angle, \angle^s	angle, angles.	\parallel^m	parallelogram.
adj.	adjacent.	\parallel^{ms}	parallelograms.
alt.	alternate.	\perp	perpendicular.
ax.	axiom.	\perp^s	perpendiculars.
\odot, \odot^s	circle, circles.	pt.	point.
const.	construction.	rect.	rectangle.
cor.	corollary.	rt. \angle	right angle.
def.	definition.	sq., sqq.	square, squares.
hyp.	hypothesis.	st.	straight.
=	is or are equal to.	\triangle, \triangle^s	triangle, triangles.
\parallel -	parallel, or is parallel to.	\therefore	therefore, or hence.

The initial letters Q.E.F. (*quod erat faciendum* = *which was to be done*) are usually placed at the end of problem to denote that the required construction has been effected.

The use of the abbreviation Q.E.D. is explained on page 66.

The foregoing are the symbols and abbreviations commonly used. Any others that may be found in the book are such as not to require explanation.

SECTION I—Angles and Triangles.

PROPOSITION 1. PROBLEM.

To describe an equilateral triangle on a given finite straight line.

Let AB be a given finite straight line. *It is required. to describe an equilateral triangle on AB .*

Construction. With centre A and radius AB , describe the $\odot BCD$; *Post.III.*

With centre B and radius BA , describe the $\odot ACE$. *Post.III.*

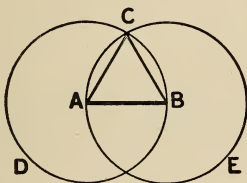


FIG. 171.

The circles must intersect; *Ax.B.*
let them intersect at C .

Join CA , CB . *Post.I.*

Then ABC is an equilateral triangle.

Proof. Because A is the centre of the $\odot BCD$,
 $\therefore AC = AB$. *Def.33.*

Again, because B is the centre of the $\odot ACE$,
 $\therefore BC = BA$, *Def.33.*

$\therefore AC = BC$. *Ax.1.*

$\therefore AB, BC, CA$ are equal to one another.

Hence $\triangle ABC$ is an equilateral \triangle ,
and it is described on AB .

Def. 48.

Q.E.F.

In Prop. 1 Euclid assumes that the circles (circumferences) cut each other, but he does not prove it. From Axiom B it is easily inferred that the circumferences of the circles intersect in two points at least. Hence the problem admits of two solutions. Illustrate by means of a diagram.

1. On the given st. line AB describe an isosceles \triangle whose equal sides are each double of AB . Three times as great as AB . Four times as great as AB .

2. Let $\triangle ABC$ be an equilat. \triangle , and let M be any point in BC . From AC cut off AN equal to CM .

3. Describe a quadrilateral having each of its sides equal to a given st. line.

4. Let $\triangle ABC$ be any \triangle . Draw a st. line equal to the sum of AB , BC , and CA .

5. Show that according to Axiom B the circles ACE , and BCD must cut each other.

PROPOSITION 2. PROBLEM.

From a given point to draw a straight line equal to a given straight line.

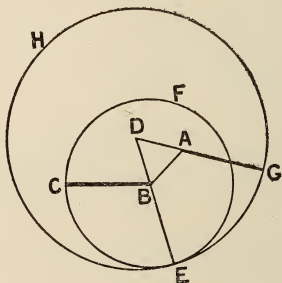


FIG. 172.

Let A be the given point, and BC the given straight line.

It is required to draw from A a straight line equal to BC .

Construction. Join AB ; *Post. I.*
and on AB describe the equilat. $\triangle DAB$. *I. 1.*

With centre B and radius BC describe the $\odot CEF$. *Post. III.*

Produce DB until it meets the $\odot CEF$ at E . *Post. II.*

With centre D and radius DE describe the $\odot EGH$.
Post. III.

Produce DA until it meets the $\odot EGH$ at G . *Post. II.*

Then AG is the straight line required.

Proof. Because DG, DE are radii of the $\odot EGH$,
 $\therefore DG = DE$, *Def. 33.*
and DA, DB , parts of them are equal, *Def. 48.*
 \therefore the remainder $AG =$ the remainder BE . *Ax. 3.*

Again because BC, BE are radii of the $\odot CEF$,
 $\therefore BC = BE$. *Def. 33.*

But $AG = BE$; *Proved.*

$\therefore AG = BC$, *Ax. 1.*
and it is drawn from the given point A . *Q.E.F.*

The student will observe that the construction of Prop. 2 may take different forms :

(i) The point A may be joined to either extremity of BC .

(ii) The equilateral triangle may be constructed on either side of AB , or of AC .

(iii) If BD (produced both ways) cuts the circumference of the circle CEF at E and K , then either DE or DK may be taken as the radius of the circle having D as its centre.

A little consideration of the foregoing will show that the problem admits of eight solutions.

1. Show how to draw from A another st. line equal to BC .
2. Let AC be joined instead of AB . Work out the solution, and prove it.
3. From A draw a st. line = twice BC .
4. On the st. line AB describe an isosceles triangle having its equal sides each equal to CD .

5. How do we know that **DA** produced as in Fig. 172 will cut the circle **EGH**?

6. In the given st. line **AB** take any two points **P** and **Q**. Draw a triangle whose sides shall be respectively equal to **AP**, **PQ**, **QB**. Is this possible in all cases?

PROPOSITION 3. PROBLEM.

From the greater of two given straight lines to cut off a part equal to the less.

Let **AB** and **C** be the two given straight lines of which **AB** is the greater.

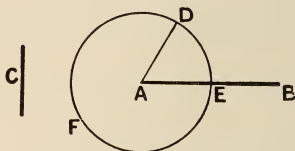


FIG. 173.

*It is required to cut off from **AB** a part equal to **C**.*

Construction. From **A** draw the st. line **AD = C**. *I.2.*

With centre **A** and radius **AD** describe the \odot **DEF** cutting **AB** at **E**. *Post.III.*

Then **AE** is the part of **AB** required.

Proof. Because **AE**, **AD** are radii of the circle **DEF**,

$\therefore \text{AE} = \text{AD}.$

Def.33.

But **C = AD**,

Const.

$\therefore \text{AE} = \text{C},$

Ax.1.

and it is cut off from **AB**.

Q.E.F.

NOTE.—According to Euclid compasses are not used for transferring distances; hence the necessity for Props. 2 and 3.

1. **AB** is less than **CD**. Produce **AB** to **E** so that **AE** may be equal to **CD**.

2. Draw a st. line equal to the sum of two given straight lines. Equal to their difference. Equal to their sum and difference taken together.

3. Construct a quadrilateral one of whose angles is the given $\angle A$ and whose sides are each equal to the given st. line CD .

4. Review Props. 1, 2, and 3, supplying the premises that are omitted in the proof of each.

PROPOSITION 4. THEOREM.

If two triangles have two sides and the included angle of the one respectively equal to two sides and the included angle of the other, then the triangles are equal in all respects.

(Proof given on pp. 100-101.)

In proving the truth of Prop. 4, the argument is as follows:

(a) *Magnitudes which can be made to coincide with one another are equal.* (Admitted without proof.)

(b) The triangles ABC , DEF can be made to coincide with each other. (Proved. See Fig. 102, page 100.)

(c) Therefore the triangles ABC , DEF are equal in the respects in which they coincide; that is, they are equal in all respects.

Show that the whole argument by which the truth of (b) is established consists of four distinct parts, each of which is made up of two propositions and a conclusion based upon them.

1. AB is a given st. line which is bisected at D . Through D a st. line PDQ is drawn \perp to AB . Show that every point in PQ is equidistant from the extremities of AB .

2. Show that the bisector of the vertical angle of an isosceles triangle will pass through the middle point of the base.

3. The straight lines drawn from the ends of the base of an isosceles triangle to the middle points of the opposite sides are equal.

4. If the st. lines PQ and RS bisect each other at right \angle any point in either of these lines is equidistant from the ends of the other.

5. $ABCD$ is a quadrilateral having the sides AB , CD equal, and also the angles ABC , DCB . Show that the diagonals AC , BD are equal.

6. Let PQR be a triangle of which the sides PQ and PR are equal. Prove the following:

(i) If equal parts PS and PT are cut off from PQ and PR respectively, the triangles PSR and PTQ are identically equal.

(ii) If PQ and PR are produced to X and Y so that $PX=PY$, the triangles PQY and PRX are identically equal; also, the triangles QRY and RQX are identically equal.

7. In the quadrilateral $ABCD$, the side $AB=AD$ and the angle BAD is bisected by the diagonal AC . Show that if the triangle ABC were rotated about AC through an angle of 180° it would coincide with the triangle ADC .

8. In the triangles ABC, DEF , let $BC=EF$, $\angle ABC=\angle DEF$, and $\angle ACB=\angle DFE$. Show that the triangles can be made to coincide. Write the argument in full.

PROPOSITION 5. THEOREM.

The angles at the base of an isosceles triangle are equal; and if the equal sides are produced, the angles on the other side of the base are also equal.

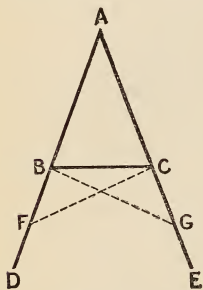


FIG. 174.

In the triangle ABC let AB be equal to AC , and let AB, AC be produced to D and E .

It is required to prove that the angles ABC, ACB are equal; also that the angles CBD, BCE are equal.

Construction. In BD take any point F , and from AE cut off a part AG equal to AF .

I.3.

Join FC, GB .

Post.1.

Proof. Then in $\triangle^s FAC, GAB$, because

$$\left\{ \begin{array}{l} FA = GA, \\ AC = AB, \\ \text{and } \angle FAC = \angle GAB; \end{array} \right.$$

Const.

Hyp.

$\therefore \triangle^s \text{FAC}, \text{GAB}$ are equal in all respects ; *I.4.*
 so that $\text{FC} = \text{GB}$, $\angle \text{ACF} = \angle \text{ABG}$, and $\angle \text{AFC} = \angle \text{AGB}$.

Again, because the wholes AF , AG are equal, *Const.*
 and the parts of these AB , AC are also equal, *Hyp.*
 \therefore the remainder $\text{BF} =$ the remainder CG . *Ax.3.*

Then in $\triangle^s \text{BFC}, \text{CGB}$, because

$$\left\{ \begin{array}{l} \text{BF} = \text{CG}, \\ \text{FC} = \text{GB}, \\ \text{and } \angle \text{BFC} = \angle \text{CGB}; \end{array} \right\} \text{Proved.}$$

$\therefore \triangle^s \text{BFC}, \text{CGB}$ are equal in all respects ; *I.4.*
 so that $\angle \text{FBC} = \angle \text{GCB}$, and $\angle \text{BCF} = \angle \text{CBG}$.

But the whole $\angle \text{ABG} =$ whole $\angle \text{ACF}$, *Proved.*
 and the parts of these, namely the $\angle^s \text{CBG}, \text{BCF}$ are also equal ;

\therefore the remaining $\angle^s \text{ABC}, \text{ACB}$ are equal ; and these are the angles at the base.

It has also been proved that $\angle^s \text{FBC}, \text{GCB}$ are equal ; and these are the angles on the other side of the base. *Q.E.D.*

COROLLARY :

An equilateral triangle is also equiangular.

1. In the figure to I. 5, join FG . Show that the triangles BFG, CGF are equal in all respects.

2. ABC, DBC are two isosceles triangles on opposite sides of the base BC . Show that AD bisects the angles at A and D .

3. If two isosceles triangles are on the same base and on the same side of it, one of these triangles must be wholly within the other.

4. If two angles of a triangle are unequal, the sides opposite to them are unequal. (Indirect proof. See page 71.)

5. The st. lines joining the middle points of the sides of an equilateral triangle are equal.

6. The straight lines joining the middle points of the sides of an isosceles triangle form an isosceles triangle.

7. If all the sides of a quadrilateral are equal, prove :

(i) That its opposite angles are equal.

(ii) That each of its diagonals bisects the angles through which it passes.

(iii) That its diagonals bisect each other at right angles.

8. Prove Prop. 5 by the method of superposition.

9. What is meant by *converse proposition*? Write the converse of, 'All horses are animals.' Give illustrations of the fact that the converse of a true proposition may be false.

10. Write the general enunciation of Prop. 5 as two propositions, and state the converse of each.

PROPOSITION 6. THEOREM.

If two angles of a triangle are equal, then the sides opposite to these angles are equal.

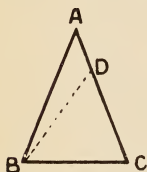


FIG. 175.

Let ABC be a triangle in which the angle ABC is equal to the angle ACB .

It is required to prove that AC is equal to AB .

Construction. Now AC, AB are either equal or unequal. Suppose that they are unequal, AC being greater than AB .

From AC cut off $CD = AB$.
and join BD .

I.3.

Post. I.

Proof. Then in $\triangle^s ABC, DCB$, because

$$\left\{ \begin{array}{l} AB = DC, \\ BC = CB, \\ \text{and } \angle ABC = \angle DCB; \end{array} \right. \quad \text{Const.}$$

*The application here made of I. 3, is likely to present some difficulty to the beginner. Observe we have made a supposition; namely, AC is greater than AB . Then in accordance with this supposition we cut off from AC a part CD equal to AB . The object of the demonstration is to show that this supposition is absurd.

\therefore the area of $\triangle ABC$ is equal to that of $\triangle DBC$, I.4.
 which is impossible, for $\triangle DBC$ is a part of $\triangle ABC$. Ax.9.

$\therefore AC$ is not greater than AB .

Similarly it may be shown that AC is not less than AB .

$\therefore AC$ is not unequal to AB , that is, $AC = AB$. Q.E.D.

COROLLARY :

An equiangular triangle is also equilateral.

1. If in the quadrilateral $KLMN$, $KL = KN$, and $\angle KLM = \angle KNM$; then $LM = NM$.

2. The \angle^s PQR , PRQ at the base of the isosceles $\triangle PQR$ are bisected by the st. lines QX , RX . Show that PX bisects the vertical $\angle QPR$.

3. In the figure to I. 5, let BG , CF intersect at O . Prove that AO is the bisector of the angle at A .

4. If two sides of a triangle are unequal, then the angles opposite to them are unequal.

5. Show that in the figure to I. 1, if AB is produced both ways it will cut the circumferences of the circles at points equidistant from C .

6. What is meant by *indirect demonstration*? What use is made of the *reductio ad absurdum* and of the *proof by exhaustion* in Prop. 6? Explain fully.

PROPOSITION 8. THEOREM.

If two triangles have the three sides of the one respectively equal to the three sides of the other, then the triangles are equal in all respects.

(Proof given on page 109.)

NOTE.—Euc. I. 7 is enunciated as follows : “On the same base and on the same side of it, there cannot be two triangles having their sides which are terminated at one extremity of the base equal to one another, and likewise those which are terminated at the other extremity equal to one another.” Todhunter remarks : “I. 7 is only required in order to lead to I. 8. The two might be superseded by another demonstration of I. 8, which has been recommended by many writers.”

Euclid enunciates I. 8 thus: "*If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them, of the other.*" Observe that the proposition in this form merely asserts that the angles opposite to the bases are equal. To establish the congruence of the triangles it is necessary to apply I. 4.

1. If a straight line is drawn from the vertex of an isosceles triangle to the middle point of the base it is perpendicular to the base, and bisects the vertical angle.

2. Two isosceles triangles ABC , DBC , stand on the same base BC . Show that AD (produced if necessary) (i) bisects both vertical angles; (ii) bisects BC at right angles. Show that every point in AD is equidistant from B and C .

3. ABC , DBC are two triangles on the same side of BC , such that $AB=DC$ and $AC=DB$; AC and DB intersect at F . Show that the triangle AFD is isosceles.

4. Two circles whose centres are P and Q intersect at X and Y . Show that PQ is perpendicular to XY , and passes through its middle point.

5. If the opposite sides of a quadrilateral are equal, then the opposite angles are equal.

6. If all the sides of a quadrilateral are equal, and if its diagonals are also equal, then all its angles are equal.

PROPOSITION 9. PROBLEM.

To bisect a given angle.

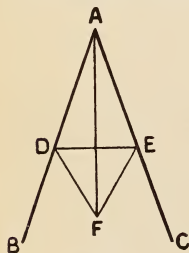


FIG. 176.

Let BAC be the given angle. *It is required to bisect the angle BAC .*

Const. In AB take any point D .
From AC cut off $AE = AD$. I.3.

Join DE , and on it, on the side remote from A , describe the equilateral $\triangle DEF$. I.1.

Join AF .

Then AF bisects the angle BAC .

Proof. In the $\triangle DAF, EAF$, because

$$\left\{ \begin{array}{l} DA = EA, \\ AF = AF, \\ \text{and } DF = EF; \end{array} \right. \quad \begin{array}{l} \text{Const.} \\ \text{Def. 48.} \end{array}$$

$\therefore \angle DAF = \angle EAF$; I. 8.

that is, $\angle BAC$ is bisected by AF . Q.E.F.

1. Show that an isosceles \triangle described on DE would serve the purpose as well as an equilateral \triangle . Why does Euclid make use of the latter rather than the former? Why is the equilateral \triangle described on the side of DE remote from A ?

2. Divide a given angle into eight equal parts.

3. Show that AF is an axis of symmetry of the figure to Prop. 9.

PROPOSITION 10. PROBLEM.

To bisect a given finite straight line.

Let AB be the given finite straight line.

It is required to bisect AB .

Const. On AB describe an equilateral $\triangle ABC$. I. 1.

Bisect $\angle ACB$ by the st. line CD meeting AB at D . I. 9.

Then AB is bisected at the point D .

Proof. In the $\triangle ACD, BCD$, because

$$\left\{ \begin{array}{l} AC = BC, \\ CD = CD, \\ \text{and } \angle ACD = \angle BCD; \end{array} \right. \quad \begin{array}{l} \text{Def. 48.} \\ \text{Const.} \end{array}$$

$\therefore AD = BD$; I. 4.

that is, the st. line AB is bisected at the point D . Q.E.F.

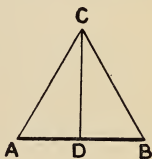


FIG. 177.

1. The bisector of the vertical angle of an isosceles triangle bisects the base.
2. Show that an angle can have only one bisector
3. Give a practical method of bisecting a given straight line. (See Problem 3, page 78.)
4. Divide a given straight line into four equal parts.
5. On AB describe an isosceles triangle ABC such that the sum of AC and BC may be equal to the given straight line PQ .
6. From a given point P draw a straight line equal to half the difference of the given straight lines AB and CD .

PROPOSITION 11. PROBLEM.

To draw a straight line at right angles to a given straight line from a given point in it.

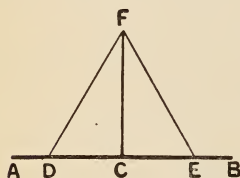


FIG. 178.

Let AB be the given st. line, and C the given point in it: *it is required to draw from C a st. line at rt. \angle^s to AB .*

Const. In AC take any point D , and from CB cut off CE equal to CD .

I.3.

On DE describe the equilateral $\triangle DEF$, and join CF .

I.1.

Then CF is a straight line drawn as required.

Proof. In the $\triangle^s DCF, ECF$, because

$$\left\{ \begin{array}{ll} DC = EC, & \text{Const.} \\ CF = CF, & \\ \text{and } DF = EF, & \text{Const.} \end{array} \right.$$

$$\therefore \angle DCF = \angle ECF;$$

I.8.

and these are adjacent angles.

$\therefore CF$ is at right angles to AB .

Def. 24.
Q.E.F.

1. Find the position of **FC** by a practical method.
2. Show that all points in **CF**, as in the figure to I. 11, are equidistant from **D** and **E**. Can any point not in **FC** (produced indefinitely both ways) be equidistant from **D** and **E**? Prove.

PROPOSITION 12. PROBLEM.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

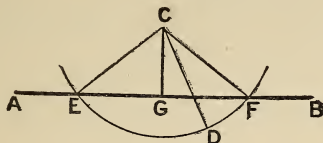


FIG. 179.

Let **AB** be the given st. line which may be produced either way, and **C** the given point without it: *it is required to draw from C a st. line \perp to AB.*

Const. On the side of **AB** remote from **C**, take any point **D**, and join **CD**.

With centre **C** and radius **CD**, describe the \odot **EDF** intersecting **AB** at **E** and **F**. *Post. III.*

Bisect **EF** at **G**. *I. 10.*

and join **CG**.

Then the st. line **CG** is \perp to **AB**.

Join **CE**, **CF**.

Proof. In the \triangle^s **EGC**, **FGC**, because

$$\left\{ \begin{array}{ll} \text{EG} = \text{FG}, & \text{Const.} \\ \text{GC} = \text{GC}, & \\ \text{and CE} = \text{CF}; & \text{Def. 33.} \end{array} \right.$$

$$\therefore \angle CGE = \angle CGF;$$

I.8.

and these are adjacent angles.

$$\therefore CG \text{ is } \perp \text{ to } AB.$$

Def. 24.

Q.E.F.

NOTE.—It is here assumed that the circumference of the circle EDF cannot cut the straight line AB at more than two points; for otherwise more than one straight line could be drawn from C \perp to AB. After reading Prop. 19, this assumption may be proved to be true.

1. Show that in accordance with Axiom B the circumference of \odot EDF must intersect AB in two points at least. Why is the point D taken on the side of AB remote from C?

2. Find the position of CG by a practical method. Compare the construction with that given by Euclid.

3. What is meant by 'the distance of a point from a straight line'?

PROPOSITION A. THEOREM.

All right angles are equal.

Let the st. line AB be at rt. \angle^s to the st. line CBD at the point B; also, let the st. line EF be at rt. \angle^s to the st. line GFH at the point F.

It is required to prove that either of the \angle^s ABC or ABD is equal to either of the \angle^s EFG or EFH.

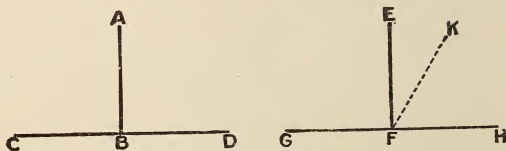


FIG. 180

Let the st. line CBD be placed so that it lies along the st. line GFH, the point B falling on the point F, and the points A and E on the same side of GFH.

BA will then fall along FE; for if not, let it take some other position as FK.

Then $\angle KFG = \angle ABC$, and $\angle KFH = \angle ABD$. Ax.8.

But $\angle ABC = \angle ABD$ by hypothesis ;

$\therefore \angle KFG = \angle KFH$.

Again, $\angle EFG = \angle EFH$, by hypothesis ;

and $\angle KFG$ is greater than $\angle EFG$; Ax.9.

$\therefore \angle KFG$ is greater than $\angle EFH$.

But $\angle EFH$ is greater than $\angle KFH$. Ax.9.

Much more then is $\angle KFG$ greater than $\angle KFH$.*

But it has been shown that $\angle^s KFG, KFH$ are equal, that is, $\angle^s KFG, KFH$ are both equal and unequal, which is absurd.

Therefore BA falls along FE , and $\angle ABC$ coincides with $\angle EFG$;

$\therefore \angle ABC = \angle EFG$.

$\therefore \angle ABC = \angle EFH$.

Hence also, $\angle ABD = \angle EFG$ or $\angle EFH$. Q.E.D.

NOTE.—The object of proving this theorem—Euclid's eleventh axiom—is not to remove any doubt as to the truth of it, for such would be unnecessary. The objection to accepting it without proof lies in the fact that it can be deduced from simpler truths.

COROLLARIES :

1. *The complements of equal angles are equal.*

2. *The supplements of equal angles are equal.*

*The argument here used is a very simple one. It may be illustrated thus : John is taller than James ; but James is taller than William. Much more, (that is, with stronger reason) then, is John taller than William.

PROPOSITION 13. THEOREM.

If a straight line stands upon another straight line, then the adjacent angles are either two right angles, or together equal to two right angles.

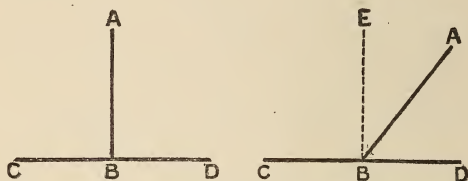


FIG. 181.

Let the st. line AB stand upon the st. line CD ; it is required to prove that $\angle^s ABC, ABD$ are either two right \angle^s , or together equal to two right \angle^s .

Proof. If $\angle ABC = \angle ABD$, they are two rt. \angle^s .

Def. 24.

If $\angle ABC$ is not $= \angle ABD$, from B draw $BE \perp$ to CD .

I. 11.

Then sum of $\angle^s ABC, ABD =$ sum of $\angle^s EBC, EBA, ABD$;
also, sum of $\angle^s EBC, EBD =$ sum of $\angle^s EBC, EBA, ABD$;
 \therefore sum of $\angle^s ABC, ABD =$ sum of $\angle^s EBC, EBD$; *Ax. 1.*

But $\angle^s EBC$ and EBD are two rt. \angle^s .

Const.

$\therefore \angle^s ABC, ABD$ are together $=$ two rt. \angle^s .

Q.E.D.

1. The exterior angles formed by producing the base of an isosceles triangle both ways are equal.

2. Show that the bisectors of the adjacent angles which one st. line makes with another are \perp to each other.

3. Show that in the figure to Prop. 13, BE is the only line that can be drawn from $B \perp$ to CD .

4. If two st. lines cut each other the four angles at the point where they cut are together equal to four right angles.

PROPOSITION 14. THEOREM.

If at a point in a straight line two other straight lines on opposite sides of it make the adjacent angles together equal to two right angles, then these two straight lines are in one and the same straight line.

(Proof given on page 69.)

1. What axioms, or truths already proved are used in the proof of Prop. 14? State the converse of this proposition.

2. If from any point X in the st. line PQ the st. lines XY and XZ be drawn on opposite sides of PQ , making the angles PXY and QXZ equal, then XY and XZ are in the same st. line.

3. $ABCD$ is a quadrilateral whose opposite sides are equal. P is the middle point of the diagonal AC . Show that PB and PD are in the same st. line.

PROPOSITION 15. THEOREM.

If two straight lines cut one another, then the vertically opposite angles are equal.

(Proof given on page 70.)

COROLLARY :

If any number of straight lines meet at a point, the sum of all the angles formed by these lines, each with the next in order, is equal to four right angles.

1. The bisectors of vertically opposite angles are in the same straight line.

2. If the diagonals of a quadrilateral bisect each other, then its opposite sides are equal.

3. To construct a rhombus when the diagonals are given.

4. Is Euclid's eleventh axiom used in the proof of Prop. 15? Write the proof in syllogistic form.

PROPOSITION 16. THEOREM.

If a side of a triangle is produced, then the exterior angle is greater than either of the interior opposite angles.

(Proof given on page 102.)

NOTE.—In the proof of Prop. 16 it is assumed that the angle ECF , as in Fig. 103, is a part of the angle ECD . This may be proved as follows:

Since FE produced meets BD at B , and EC, CF meet it at C , therefore no side of the triangle ECF can meet BD at any other point (Ax. 10). Now if any point in CD were within the triangle ECF , CD (produced if necessary) would cut at least one of its sides at another point (Ax. B). But this is shown to be impossible; hence, every point in CD lies without the triangle ECF .

1. The sum of the exterior angles formed by producing a side of a triangle both ways is greater than two right angles.

2. Only one perpendicular can be drawn to a straight line from a point without it.

PROPOSITION 17. THEOREM.

Any two angles of a triangle are together less than two right angles.

(See page 103.)

1. A triangle must have at least two acute angles.

2. The three angles of an equilateral triangle are acute.

3. In the $\triangle ABC$ let the $\angle ACB$ be obtuse. Show that the \perp drawn from A to BC must meet BC produced.

PROPOSITION 18. THEOREM.

If two sides of a triangle are unequal, then the opposite angles are unequal, and the greater side has opposite to it the greater angle.

(Proof given on page 104.)

NOTE.—Euclid's enunciation of Prop. 18 is as follows: *The greater side of every triangle has the greater angle opposite to it.*

1. All the \angle s of a scalene \triangle are unequal.
 2. PQRS is a quadrilateral of which the shortest side is PQ and the longest SR. Show that \angle QRS is less than \angle QPS.
-

PROPOSITION 19. THEOREM.

If two angles of a triangle are unequal, then the opposite sides are unequal, and the greater angle has opposite to it the greater side.

(Proof given on page 106.)

NOTE.—Euclid's enunciation of Prop. 19 is as follows: *The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.*

1. In a right-angled triangle the hypotenuse is the greatest side.
 2. In the $\triangle ABC$ the $\angle B$ is bisected by the st. line BD which meets AC at D. Show that AB is greater than AD, and BC greater than CD.
 3. A straight line joining the vertex of an isosceles triangle to any point in the base produced is greater than either of the equal sides.
 4. *The perpendicular is the shortest of all the straight lines that can be drawn from a given point to a given straight line; and, of any other two, the one that makes the smaller angle with the perpendicular is the shorter.*
-

PROPOSITION 20. THEOREM.

Any two sides of a triangle are together greater than the third side.

(Proof given on page 107.)

1. Prove Prop. 20 by drawing a \perp from the vertex to the base.
2. *Any side of a triangle is greater than the difference of the other two sides.*

3. The sum of any three sides of a quadrilateral is greater than the fourth side.

4. The sum of the diagonals of a quadrilateral is greater than the sum of either pair of opposite sides.

5. The sum of the sides of a quadrilateral is greater than the sum, and less than twice the sum of its diagonals.

6. *The diameter is the greatest chord in a circle.*

7. *The sum of the two sides of a triangle is greater than twice the straight line drawn from the vertex to the middle point of the base.*

8. Let PQ be a given st. line and X and Y be two given points without it. Find in PQ (produced if necessary) a point Z such that XZ and YZ will make equal angles with PQ .

Show that the sum of XZ and YZ is less than that of any other two straight lines drawn from X and Y to a point in PQ .

PROPOSITION 21. THEOREM.

If from the ends of a side of a triangle two straight lines are drawn to any point within the triangle, these straight lines are together less than the other two sides of the triangle, but they contain a greater angle.

Let ABC be a triangle, and let the two straight lines BD and CD be drawn from the end-points of the side BC to any point D within the triangle.

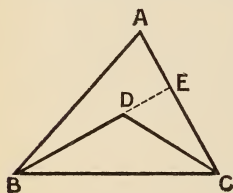


FIG. 182.

It is required to prove that (i) BD and DC are together less than BA and AC , (ii) the angle BDC is greater than the angle BAC .

Construction. Produce BD to meet AC at E .

Proof. In the $\triangle BAE$, the two sides BA , AE are together greater than the third side BE ; I.20.

to each of these unequals add EC ;

then BA , AC are together greater than BE , EC .

Ax.4.

Again, DE , EC are together greater than DC .

to each of these unequals add BD ;

then BE , EC are together greater than BD , DC . *Ax.4.*

But BA , AC are together greater than BE , EC ; *Proved.*
much more then are BA , AC together greater than BD , DC .

(ii) Because the ext. $\angle BDC$ of the $\triangle DEC$ is greater than the int. opp. $\angle DEC$; *I.16.*

and because the ext. $\angle DEC$ of $\triangle BAE$ is greater than the int. opp. $\angle BAE$, that is, than $\angle BAC$.

much more therefore is the $\angle BDC$ greater than the $\angle BAC$.
Q.E.D.

1. The sum of the three straight lines drawn from the vertices of a triangle to a point within it is less than the perimeter, and greater than half the perimeter of the triangle.

2. Let a triangle and a quadrilateral stand on the same base and on the same side of it :

(i) If the quadrilateral is wholly within the triangle, then the perimeter of the quadrilateral is less than that of the triangle.

(ii) If the triangle is wholly within the quadrilateral, then the perimeter of the triangle is less than that of the quadrilateral.

3. If two triangles are on the same base and have equal vertical angles, then the vertex of each triangle lies without the other.

Loci.

1. *To find the locus of a point at a given distance from a given point.*

2. Find the locus of a point at the distance d from A , one of the extremities of the straight line AB .

3. In the given straight line LM find a point at the distance a from the given point P .

4. Find a point equidistant from two given points A and B . Find another. Join the two points thus formed. Prove that the straight line thus drawn (produced if necessary) bisects AB and is perpendicular to it.

5. Prove the following :

(a) *Through a given point in a given straight line only one perpendicular to the line can be drawn in a plane.*

(b) *Every point in the perpendicular which passes through the middle point of a given straight line is equidistant from its extremities.*

(c) *Every point not in the perpendicular which passes through the middle point of a given straight line is unequally distant from the extremities of the line.*

6. On the base **AB** construct an isosceles triangle whose height shall be equal to the given straight line **LM**.

7. In the given straight line **AB** find a point equidistant from the given points **P** and **Q**.

8. Find a point equidistant from the given points **A** and **B** and at the distance d from the given point **P**.

9. **AB** and **CD** are two given straight lines. Find a point at the distance **AB** from **A** and also at the distance **CD** from **C**. How many points satisfy both conditions?

PROPOSITION 22. PROBLEM.

To construct a triangle having its sides respectively equal to three given straight lines, any two of which are together greater than the third.

Let **A**, **B**, **C** be the three given straight lines, any two of which are together greater than the third. It is required

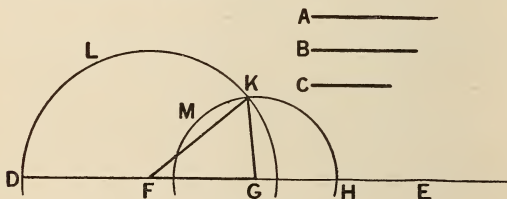


FIG. 183.

*to construct a triangle of which the sides shall be respectively equal to **A**, **B**, **C**.*

Construction. Take a straight line DE terminated at D , but unlimited towards E ;

Make $DF = A$, $FG = B$, and $GH = C$. *I.3.*

With centre F and radius FD describe $\odot DLK$.

With centre G and radius GH describe $\odot HKM$,
cutting $\odot DLK$ at K . Join FK , GK .

Then KFG is the triangle required.

Proof. Because F is the centre of $\odot DLK$,

$\therefore FK = FD$. *Def.33.*

But $FD = A$; *Const.*

$\therefore FK = A$.

Again, because G is the centre of $\odot HKM$,

$\therefore GK = GH$. *Def.33.*

But $GH = C$; *Const.*

$\therefore GK = C$.

And $FG = B$; *Const.*

$\therefore \triangle KFG$ has its sides KF , FG , GK respectively equal to the three given straight lines. *Q.E.F.*

NOTE.—In Prop. 20, Euclid proves that any two sides of a triangle are together greater than the third side. In Prop. 22 he assumes the possibility of constructing a triangle in accordance with this condition.

The truth of the assumption may be demonstrated thus: Of the three given st. lines A , B , and C , as in Fig. 183, let A be not less than either of the others. Then, by I. 20, the circle DLK must cut DE between G and H ; also, the circle HKM must cut DE between G and D . Hence there are two points on the circumference of the circle HKM such that one is *without*, and the other *within* the circle DLK ; therefore, the circumferences of these circles cut each other (Axiom B).

1. How is Prop. 1 related to Prop. 22?

2. To construct a right-angled triangle when the hypotenuse and one side are given.

3. To construct a quadrilateral whose sides shall be respectively equal to those of a given quadrilateral.

PROPOSITION 23. PROBLEM.

At a given point in a given straight line to make an angle equal to a given angle.

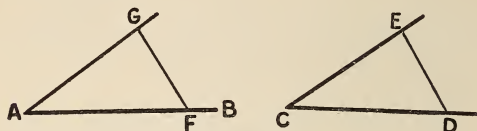


FIG. 184.

Let AB be the given straight line, A the given point in it, and DCE the given angle. *It is required to make at A an angle $= \angle DCE$.*

Const. In CD , CE , take any points D , E , and join DE .

From AB cut off $AF = CD$; I.3.

and on AF construct $\triangle AFG$ so that $FG = DE$, $GA = EC$.

I.22.

Then $\angle FAG$ is the angle required.

Proof. In the $\triangle^s FAG, DCE$, because

$$\left\{ \begin{array}{ll} FA = DC, & \text{Const.} \\ AG = CE, & \text{Const.} \\ \text{and } FG = DE; & \text{Const.} \end{array} \right.$$

$\therefore \angle FAG = \angle DCE$. I.8.

Q.E.F.

1. Make an angle twice as great as a given angle.
2. Make an angle (i) equal to the sum of two given angles ; (ii) equal to the difference of two given angles.
3. Make an angle equal to $\frac{3}{2}$ of a given angle.
4. Construct a triangle :
 - (i) When the lengths of two sides and the included angle are given.
 - (ii) When the base and the angles at the base are given.
 - (iii) When the base, an angle at the base, and the sum of the other two sides are given.

PROPOSITION 24. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, respectively, but the included angles unequal, then the base of the one which has the greater angle is greater than the base of the other.

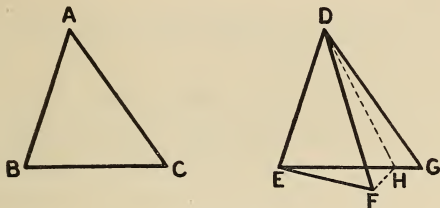


FIG. 185.

Let ABC and DEF be two triangles having AB equal to DE , AC equal to DF , but the angle BAC greater than the angle EDF .

It is required to prove that the base BC is greater than the base EF .

Proof. Suppose the $\triangle ABC$ to be applied to the $\triangle DEF$ so that A may be on D , and AB may fall along DE , then because AB is equal to ED , *Hyp.*
therefore B will fall on E .

Also, because $\angle BAC$ is greater than $\angle EDF$, *Hyp.*
therefore AC will fall on the side of DF remote from DE .
Let AC take the position of DG ; then BC will take the position of EG .

CASE I.—If EG falls along EF then EF is a part of EG ; therefore EG is greater than EF . *Ax. 9.*

But EG is equal to BC , therefore BC is greater than EF .

CASE II.—If EG does not fall along EF let the angle FDG be bisected by the straight line DH meeting EG in H . Join FH .

Then in \triangle^s FDH, GDH, because

$$\left\{ \begin{array}{ll} \text{FD} = \text{GD}, & \text{Hyp.} \\ \text{DH} = \text{DH}, & \\ \text{and } \angle \text{FDH} = \angle \text{GDH}, & \text{Const.} \end{array} \right.$$

$\therefore \text{FH} = \text{GH}.$ I.4.

But the sum of EH and FH is greater than EF; I.20.

$\therefore \text{EG}$ is greater than EF,

that is, the base BC is greater than the base EF. Q.E.D.

1. Show that by increasing the angle between the legs of a pair of compasses the distance between the points is increased.

2. PQR is a triangle and X is the middle point of QR. Show that PQ is greater than, equal to, or less than PR, according as the angle PXQ is greater than, equal to, or less than the angle PXR.

3. Write the premises not expressed in the proof of I.24.

4. How are Props. 4 and 24 related to each other?

5. Give a proof of Prop. 8 based on Props. 4 and 24.

PROPOSITION 25. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, respectively, but the bases unequal, then the angle included by the two sides of the one which has the greater base is greater than the angle included by the two sides of the other.

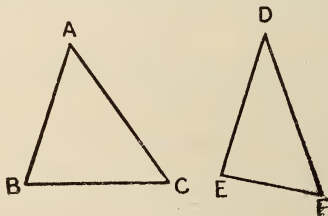


FIG. 186.

Let ABC and DEF be two triangles having AB equal to DE , AC equal to DF , but the base BC greater than the base EF .

It is required to prove that the angle BAC is greater than the angle EDF .

Proof. The $\angle BAC$ must be greater than, equal to, or less than the $\angle EDF$.

If the $\angle BAC$ were equal to the $\angle EDF$, then the base BC would be equal to the base EF ; I.4.

but BC is greater than EF , Hyp.

\therefore the $\angle BAC$ is not equal to the $\angle EDF$.

Again, if the $\angle BAC$ were less than the $\angle EDF$, then the base BC would be less than the base EF ; I.24.

but BC is greater than EF , Hyp.

\therefore the $\angle BAC$ is not less than the $\angle EDF$.

Therefore the $\angle BAC$ is greater than the $\angle EDF$. Q.E.D.

1. $PQRS$ is a quadrilateral having the adjacent sides PQ and PS equal.

(i) If the side QR is greater than the side SR , then the angle QPR is greater than the angle SPR .

(ii) If the side QR is equal to the side SR , then the angle QPR is equal to the angle SPR .

(iii) If the side QR is less than the side SR ; then the angle QPR is less than the angle SPR .

2. Write the converse of each of the foregoing (No. 1), and prove it.

3. PQR is an isosceles triangle, of which the equal sides are PQ and PR . X is any point in QR . Show that QX is greater than, equal to, or less than RX according as the angle QPX is (given) greater than, equal to, or less than the angle RPX .

4. Write the converse of No. 3, and prove it.

5. Compare the proof of I. 25 with that of I. 19.

PROPOSITION 26. THEOREM.

I. *If two triangles have two angles of the one respectively equal to two angles of the other, and also the side adjacent to the angles in the one equal to the side adjacent to the angles in the other ; then the triangles are equal in all respects.*

II. *If two triangles have two angles of the one respectively equal to two angles of the other, and also a side of one equal to a side of the other, these sides being opposite to a pair of equal angles ; then the triangles are equal in all respects.*

(See pp. 110-111.)

NOTE.—Euclid's enunciation of this proposition is as follows : *If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, viz., either the sides adjacent to the equal angles in each, or the sides opposite to them ; then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.*

The truth that a triangle is determined when the base and the angles at the base are known was discovered by Thales. In all probability it was used in finding the distance of a ship at sea.

1. The angle LMN is bisected by the straight line MO ; LON is perpendicular to MO . Show that $\text{ML} = \text{MN}$.

2. The straight lines AB and CD intersect at E . Through any point F draw a straight line cutting them at G and H so that EG and EH shall be equal.

3. Through any point P draw a straight line equidistant from the given points A and B .

4. The diagonals of a quadrilateral bisect each other if its opposite sides are equal.

5. Show how to apply Prop. 26 (I.) in finding the length of a straight line of which only one extremity is accessible.

6. Prove I.26 by the method of superposition.

PROPOSITION B. THEOREM.

If two triangles have two sides of the one respectively equal to two sides of the other, and also the angles opposite to a pair of equal sides equal; then the angles opposite to the other pair of equal sides are either equal or supplementary; and if equal, the triangles are equal in all respects.

Let $\triangle ABC$ and $\triangle DEF$ be two triangles having AB equal to DE , AC equal to DF , and the angle ABC equal to the angle DEF .

It is required to prove that the angles ACB and DFE are either equal or supplementary, and that when these angles are equal, the triangles are equal in all respects.

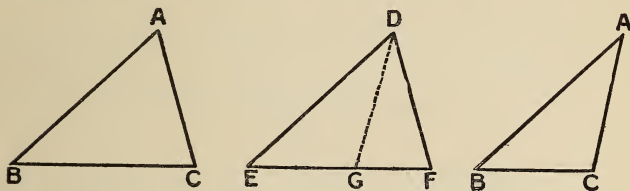


FIG. 187.

Proof. Let $\triangle ABC$ be applied to $\triangle DEF$ so that A may be on D , and AB may fall along DE ,

then because $AB = DE$,

Hyp.

$\therefore B$ will fall on E ;

Also, because $\angle ABC = \angle DEF$,

Hyp.

BC will fall along EF , and the point C will fall on EF , or EF produced.

Now BC is either equal or unequal to EF .

CASE I. If BC is equal to EF , then $\triangle^s ABC, DEF$ coincide;

$\therefore \angle ACB = \angle DFE$, and the triangles are equal in all respects.

CASE II. If BC is unequal to EF , let the point C take the position of G , and join DG ; then AC will coincide with DG , and $\angle ACB$ with $\angle DGE$.

Because $AC = DF$,

Hyp.

$\therefore DG = DF$.

Ax 1.

$\therefore \angle DFE = \angle DGF$.

I.5.

But $\angle DGE$ is supplementary to $\angle DGF$.

I.13.

$\therefore \angle DGE$ is supplementary to $\angle DFE$;

Hence $\angle ACB$ is supplementary to $\angle DFE$.

Q.E.D.

COROLLARIES :

1. *If two triangles have two sides of the one equal to two sides of the other, respectively, and have likewise the angles opposite a pair of equal sides equal, then, if the angles opposite the other pair of equal sides are both acute, both obtuse, or if one of them is a right angle, the triangles are equal in all respects.*

For if the angles ACB and DFE are both acute, or both obtuse, they cannot be supplementary; therefore, according to Prop. A they are equal, and the triangles are equal in all respects.

Again, if one of these angles, say ACB , is a right angle, then according to Prop. A the angle DFE must be a right angle; therefore the triangles are equal in all respects.

2. *Hence, if two right-angled triangles have a side and the hypotenuse of the one, respectively, equal to a side and the hypotenuse of the other, then the triangles are equal in all respects.**

For if the given angles ABC and DEF are right angles, then the angles ACB and DFE are both acute; therefore, according to Cor. 1 the triangles are equal in all respects.

* This important theorem is proved independently on page 112.

SECTION II—Parallel Straight Lines and Parallelograms.

PROPOSITION 27. THEOREM.

If a straight line intersects two other straight lines so as to make a pair of alternate angles equal, then these two straight lines are parallel.

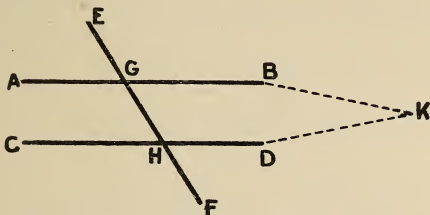


FIG. 188.

Let the straight line EF intersect the two straight lines AB , CD at G and H , so as to make the alternate angles AGH , GHD equal: *it is required to prove that AB , CD are parallel.*

Proof. For if AB and CD be not parallel, they will meet, if produced far enough either towards A and C , or towards B and D . Suppose that they meet when produced towards B and D , at the point K .

Then because KGH is a triangle,
 \therefore the ext. $\angle AGH$ is greater than the int. opp. $\angle GHD$. *I.16.*

But $\angle AGH = \angle GHD$. *Hyp.*
 $\therefore \angle s AGH, GHD$ are both equal and unequal,

which is impossible ;

\therefore AB , CD cannot meet when produced towards B and D .

In a similar manner it may be shown that AB , CD cannot meet on being produced towards A and C ;

\therefore AB is \parallel to CD .

Q.E.D.

PROPOSITION 28. THEOREM.

If a straight line intersects two other straight lines so as to make (i) a pair of corresponding angles equal, or (ii) a pair of interior angles on the same side of the line supplementary ; then, in either case, these two straight lines are parallel.

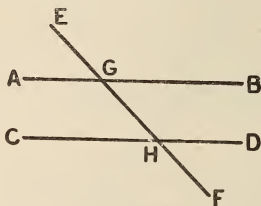


FIG. 189.

(i) Let the straight line EF intersect the two straight lines AB , CD so as to make the corresponding angles EGB and GHD equal : *it is required to prove that AB is \parallel to CD .*

Proof. Because $\angle EGB = \angle GHD$.

Hyp.

and $\angle EGB =$ vertically opp. $\angle AGH$;

I.15

$\therefore \angle AGH = \angle GHD$;

and these are alternate angles ;

$\therefore AB$ is \parallel to CD .

I.27.

(ii) Let the straight line EF intersect the two straight lines AB , CD so as to make the two interior angles BGH and GHD supplementary : *it is required to prove that AB is \parallel to CD .*

Proof. Because sum of \angle^s BGH, GHD = two rt. \angle^s ,

Hyp.

and sum of \angle^s BGH, AGH = two rt. \angle^s ;

I.13.

\therefore sum of \angle^s BGH, AGH = sum of \angle^s BGH, GHD.

From each of these equals take \angle BGH;

then the remaining \angle AGH = the remaining \angle GHD;

and these are alternate angles;

\therefore AB is \parallel to CD.

I.27.

Q.E.D.

1. Deduce I.28, directly from I.16, and I.17.

2. The straight lines AB and CD bisect each other at E. Prove that AC and BD are parallel.

3. If two straight lines in the same plane are perpendicular to the same straight line, they are parallel.

4. If a transversal of two straight lines makes the two exterior angles on the same side supplementary, the two straight lines are parallel.

5. A quadrilateral whose sides are equal is a parallelogram.

PROPOSITION 29. THEOREM.

If a straight line intersects two parallel straight lines it makes (i) the alternate angles equal; (ii) the corresponding angles equal; and (iii) the interior angles on the same side supplementary.

(Proof given on pp. 122-123.)

COROLLARY:

*If a transversal of two straight lines makes the two interior angles on one side of it together less than two right angles, the two straight lines will, if sufficiently produced, meet on that side.**

*Observe that this theorem—Euclid's twelfth axiom—is the converse of Prop. 17. This will be seen more easily by enunciating the latter thus: *The two interior angles which a transversal makes with two intersecting straight lines are together less than two right angles.*

The two st. lines cannot be \parallel , for if they were, the two interior \angle^s on one side of the transversal would be together = two rt. \angle^s , which is not the case. Hence the two st. lines will meet if sufficiently produced.

Again, the two st. lines cannot meet on the other side of the transversal, for then a triangle having two of its angles together greater than two right angles would be formed, which is impossible. Hence the conclusion.

Write the proof in full.

1. A straight line parallel to the base of an isosceles triangle makes equal angles with the sides, or the sides produced.

2. If the bisector of an exterior angle of a triangle is parallel to one of the sides, the triangle is isosceles.

3. A straight line which is perpendicular to one of any number of parallel straight lines, is perpendicular to all of them.

4. Two angles whose arms are parallel, respectively, are either equal or supplementary.

5. All the angles of a rectangle are right angles.

PROPOSITION 30. THEOREM.

Straight lines which are parallel to the same straight line are parallel to one another.

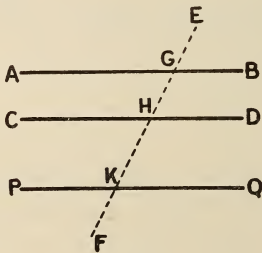


FIG. 190.

Let each of the straight lines AB and CD be parallel to the straight line PQ : it is required to prove that AB is parallel to CD .

Const. Draw any straight line EF intersecting AB , CD , and EF in the points G , H , and K , respectively.

Proof. Because AB is \parallel to PQ . *Hyp.*
 $\therefore \angle AGH = \text{alternate } \angle HKQ$. *I.29.*

Again, because CD is \parallel to PQ . *Hyp.*
 $\therefore \angle CHK = \text{alternate } \angle HKQ$. *I.29.*

$\therefore \angle AGH = \angle CHK$;
 and these are corresponding angles ;
 $\therefore AB$ is \parallel to CD .

I.28.
Q.E.D.

1. Deduce Prop. 30 from Axiom 11. (Indirect proof.)
2. If a straight line intersects any one of a series of parallel straight lines it intersects all of them.
3. If a straight line is parallel to any one of a series of parallel straight lines, it is parallel to all of them.
4. If a straight line is perpendicular to any one of a series of parallel straight lines, it is perpendicular to all of them.

PROPOSITION 31. PROBLEM.

To draw through a given point a straight line parallel to a given straight line.

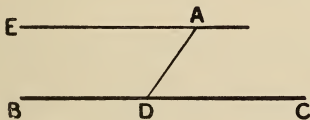


FIG. 191.

Let A be the given point, and BC the given straight line :
it is required to draw through A a straight line parallel to BC .

Const. In BC take any point D , and join AD . At the point A make $\angle DAE = \angle ADC$, and alternate to it. *I.23.*

Then EA is \parallel to BC .

Proof. Because $\angle EAD =$ alternate ADC .

Const.

$\therefore EA$ is \parallel to BC ;

I.27.

and it is drawn through the given point A .

Q.E.F.

1. Through a given point draw a straight line making with a given straight line an angle equal to a given angle.

2. Draw through a given point a straight line, of which the part intercepted by two given parallel straight lines shall be of given length.

3. Through a given point between two intersecting straight lines draw a straight line, such that the part of it intercepted by these two straight lines shall be bisected at the given point.

4. To construct a triangle when one angle, the side opposite to it, and the sum of the other two sides are given.

PROPOSITION 32. THEOREM.

Any exterior angle of a triangle is equal to the sum of the two interior opposite angles ; also, the sum of the three interior angles of a triangle is equal to two right angles.

(Proof given on page 124.)

COROLLARIES :*

1. *All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*

By joining the vertices of a rectilineal figure to a point within it, the figure is divided into as many triangles as the figure has sides. Now the sum of all the angles of these triangles is equal to twice as many right angles as the figure has sides. But all the angles of the triangles make up the interior angles of the figure, and the angles at the point within it. Hence the conclusion.

Write the proof in full. Prove by drawing diagonals from one of the vertices of the rectilineal figure.

* These corollaries were added by Robert Simson, a Scotchman, who was born in 1687 and died in 1768. His *Elements of Euclid*, issued in 1758, forms the basis of most of the modern works on Euclid.

2. *All the exterior angles of any rectilineal figure, made by producing the sides successively in the same direction, are together equal to four right angles.*

For every side there is an exterior angle and also an interior angle; therefore all the exterior angles, together with all the interior angles, are equal to twice as many right angles as the figure has sides. But the interior angles, together with four right angles, are equal to twice as many right angles as the figure has sides. Hence the conclusion.

Write the proof in full.

1. *If two triangles have two angles of the one, respectively, equal to two angles of the other, the third angle of the one is equal to the third angle of the other.*

2. *In a right-angled triangle the acute angles are complementary.*

3. *Each angle of an equilateral triangle is two-thirds of a right angle.*

4. *The sum of the angles of a quadrilateral is equal to four right angles.*

5. The sum of the exterior \angle^s of a hexagon is half the sum of the interior \angle^s .

6. PQR is an isosceles \triangle , of which P is the vertex. If QP is produced to S so that $PS = PQ$, prove that $\triangle QRS$ is right-angled.

7. *The st. line joining the middle point of the hypotenuse of a right-angled triangle to the vertex of the rt. angle is equal to half the hypotenuse.*

8. Trisect a right-angle.

9. The bisectors of two adjacent angles of a \parallel^m are \perp to each other.

10. Construct a right-angled triangle when the hypotenuse and the sum of the two sides are given.

11. The bisectors of the exterior angles at the base of a triangle form an angle equal to half the sum of the angles at the base.

12. ABC is a right-angled \triangle , of which the \angle at A is the rt. \angle . AD is drawn \perp to the hypotenuse, meeting it at D . Show that $\triangle^s ABC, ABD$ are equiangular to one another.

13. To construct a \triangle when the three \angle^s are given. Are the given conditions sufficient to determine the triangle? To what extent does I.32 afford an explanation of this?

PROPOSITION 33. THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are themselves equal and parallel.

(Proof similar to that given on page 131.)

1. The straight lines joining the extremities of two equal and parallel straight lines, towards opposite parts, bisect each other.

2. If the \perp^s drawn from the points P and Q to the straight line AB are equal, show that PQ must be either \parallel to AB or bisected by AB .

3. If the straight lines joining the adjacent extremities of two equal straight lines are equal, then they are parallel.

4. *The straight line joining the middle points of any two sides of a triangle is parallel to third side, and equal to half of it.*

PROPOSITION 34. THEOREM.

The opposite sides and angles of a parallelogram are equal, and either diagonal bisects it.

(Proof given on page 132.)

1. If one angle of a parallelogram is a right angle, all its angles are right angles.

2. If both pairs of opposite sides of a quadrilateral are equal, the figure is a parallelogram.

3. If both pairs of opposite angles of a quadrilateral are equal, the figure is a parallelogram.

4. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

5. *The diagonals of a parallelogram bisect each other.*

6. If the diagonals of a \parallel^m are equal the figure is a rectangle.

7. If two adjacent sides of a \parallel^m are equal, its diagonals are \perp to each other.

8. If a straight line passes through the middle point of a diagonal of a parallelogram, the portion of it intercepted by a pair of opposite sides is bisected at that point.

9. Every straight line drawn through the middle point of a diagonal of a parallelogram divides the parallelogram into two equal parts.

10. Through a given point draw a straight line which shall bisect a given parallelogram.

11. *If two parallelograms have two adjacent sides and the included angle of the one respectively equal to two adjacent sides and the included angle of the other, then the parallelograms are identically equal. (Superposition.)*

Loci.

1. *Find the locus of a point at a given distance from a given straight line.*

2. *Find the locus of a point equidistant from two given parallel straight lines.*

3. *Find the locus of a point equidistant from two intersecting straight lines.*

4. Find the locus of the vertices of all right-angled triangles having a common hypotenuse. (Apply Ex. 7, page 215.)

5. A ladder slides downwards along a vertical wall against which it leans, the foot moving in a direction \perp to the wall. Find the locus of the middle point of the ladder.

6. To construct a rectangle when one side and a diagonal are given.

7. To construct a rectangle when a diagonal and the angle formed by it and one side are given.

8. To construct a \parallel^m when both diagonals and the angle formed by them are given.

9. Find the locus of the middle points of straight lines drawn from a given point to the circumference of a given circle. (Apply Ex. 4, middle of page 216.)

SECTION III—Areas of Parallelograms, Triangles, and Squares.

PROPOSITION 35. THEOREM.

Parallelograms on the same base and between the same parallels are equal in area.

(Proof given on page 141.)

1. Construct a rectangle equal in area to a given \parallel^m .
2. Construct a rhombus on the same base and between the same \parallel^s as a given \parallel^m .
3. What is meant by the *altitude of a parallelogram*? Show that \parallel^m s of unequal altitude on equal bases are unequal in area.
4. Equal \parallel^m s on a common base, and on the same side of it, are between the same \parallel^s .

PROPOSITION 36. THEOREM.

Parallelograms on equal bases and between the same parallels are equal in area.

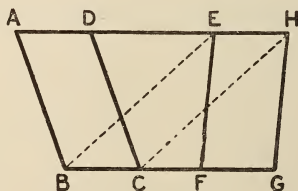


FIG. 192.

Let the \parallel^m s $ABCD$, $EFGH$ be on equal bases BC , FG , and between the same \parallel^s AH , BG .

It is required to prove that \parallel^m s $ABCD$, $EFGH$ are equal in area.

Construction. Join BE, CH.

Proof. Write the proof.

The argument is as follows :

(a) Things which are equal to the same thing are equal to one another.

(b) The \parallel^{ms} ABCD, EFGH are each equal to the \parallel^{m} EBCH.

(c) Therefore \parallel^{m} ABCD = \parallel^{m} EFGH.

Before (b) can be admitted into the argument, it must be shown (i) that EBCH is a \parallel^{m} ; (ii) that the \parallel^{ms} ABCD, EFGH are each equal to the \parallel^{m} EBCH.

1. Show how to divide a \parallel^{m} into two equal \parallel^{ms} .
2. *Parallelograms on equal bases and of equal altitudes are equal in area.*
3. *Two rectangles are identically equal, if they have two adjacent sides of the one respectively equal to two adjacent sides of the other.*
4. Of two parallelograms on equal bases, the one having the greater altitude is the greater. State and prove the converse of this theorem.
5. Describe a parallelogram three times as great as a given parallelogram.

PROPOSITION 37. THEOREM.

Triangles on the same base and between the same parallels are equal in area.

(Proof given on page 142.)

1. *Triangles on the same base and of equal altitudes are equal in area.*
2. PQRS is a \parallel^{m} and X is any point in PS. Show that the sum of the Δ^{s} PQX, SRX = half the \parallel^{m} .
3. P is any point in the diagonal BD of the \parallel ABCD. Show that the Δ^{s} PAB, PCB are equal in area.
4. PQRS is a \parallel^{m} and X is any point within it. Show that the sum of the Δ^{s} PQX, SRX = the sum of the Δ^{s} PSX, QRX.
5. On the base AB of the Δ ABC describe an equivalent Δ having its vertex in the given straight line PQ. Discuss impossible cases.
6. Show how to transform any polygon into an equivalent Δ . (See page 138.)
7. Transform the pentagon ABCDE into an equivalent rectangle.

PROPOSITION 38. THEOREM.

Triangles on equal bases and between the same parallels are equal in area.

Let ABC , DEF be triangles on equal bases BC , EF , and between the same parallels AD , BF .

It is required to prove that $\triangle^s ABC$, DEF are equal in area.

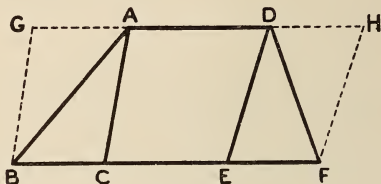


FIG. 193.

Const. Through B draw $BG \parallel$ to CA meeting DA produced at G , and through F draw $FH \parallel$ to ED meeting AD produced at H . *I.31.*

Proof. Write the proof.

The argument is as follows :

(a) Things which are halves of equal things are equal to one another.

(b) The triangles ABC , DEF are respectively halves of the equal \parallel^{ms} $GBCA$, $DEFH$.

(c) Therefore the triangles ABC , DEF are equal in area.

To establish the truth of (b) it is necessary to show (i) that $GBCA$, $DEFH$ are equal \parallel^{ms} (ii) that the $\triangle^s ABC$, DEF are respectively halves of the \parallel^{ms} $GBCA$, $DEFH$.

1. *Triangles on equal bases and of equal altitude are equal in area.*

2. *Each of the medians of a triangle divides the triangle into two equal parts.*

3. X is any point in the median AD of the triangle ABC . Prove that the triangles ABX , ACX are equal in area.

4. ABC is a triangle, and D , E , F , are the middle points of its sides. Show that the triangle DEF is one-fourth of the triangle ABC .

5. The sides LM , LN of the triangle LMN are bisected at P , Q . If MQ , NP intersect at R , show that the quadrilateral $LPRQ$ is equivalent to the triangle MNR .

6. The diagonals of a \parallel^m divide it into four equal triangles.

7. Let KLM be any \triangle , and let P and Q be the middle points of KL and LM . Join MP and KQ , and let them intersect at X . Show that $\triangle KXM$ is one-third of $\triangle KLM$.

8. To divide a triangle into two equal parts by a st. line drawn through a given point in one of the sides.

PROPOSITION 39. PROBLEM.

Equal triangles on the same base and on the same side of it are between the same parallels.

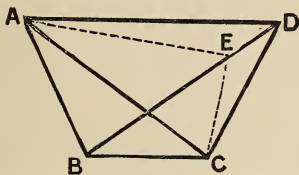


FIG. 194.

Let the equal $\triangle^s ABC$, DBC be on the same base BC , and on the same side of it; also, let A , D be joined.

It is required to prove that AD is \parallel to BC .

Const. If AD be not \parallel to BC , through A draw $AE \parallel$ to BC ,
I.31.

meeting BD , or BD produced at E , and join EC .

Proof. Then because $\triangle^s ABC$, EBC are on the same base and between the same \parallel^s ,

$$\therefore \triangle ABC = \triangle EBC. \quad I.37.$$

$$\text{But} \quad \triangle ABC = \triangle DBC; \quad Hyp.$$

$$\therefore \triangle EBC = \triangle DBC,$$

that is, the part = the whole, which is impossible. Ax.9.

$\therefore AD$ is \parallel to BC . Q.E.D.

1. $ABCD$ is a quadrilateral whose diagonals AC , BD intersect at O . If the triangles AOB , DOC are equal in area, then AD is parallel to BC .

2. The st. line drawn through the middle point of one side of a triangle parallel to a second side bisects the third side.

State the converse of this theorem.

3. The middle points of the sides of a quadrilateral are the vertices of a \parallel^m .

4. Two triangles equal in area are on opposite sides of a common base. Show that the middle point of the st. line joining their vertices lies in the base or the base produced.

5. Of all the equal triangles that can stand on the same base the isosceles triangle has the least perimeter.

PROPOSITION 40. THEOREM.

Equal triangles on equal bases in the same straight line and on the same side of it are between the same parallels.

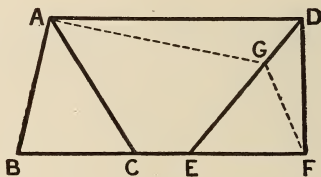


FIG. 195.

Let the equal $\triangle^s ABC$, DEF be on equal bases BC , EF in the same straight line BF and on the same side of it; also, let AD be joined.

It is required to prove that AD is \parallel to BF .

Const. If AD be not \parallel to BF draw $AG \parallel$ to BF , *I.31*, meeting ED , or ED produced at G , and join GF .

Proof. Then because $\triangle^s ABC, GEF$ are on equal bases and between the same \parallel^s ,

$$\therefore \triangle ABC = \triangle GEF. \quad I.38.$$

$$\text{But} \quad \triangle ABC = \triangle DEF; \quad \text{Hyp.}$$

$$\therefore \triangle DEF = \triangle GEF;$$

that is, a part = the whole, which is impossible. Ax.9.

$\therefore AD$ is \parallel to BF . Q.E.D.

1. *Equal triangles on equal bases have equal altitudes.*

2. *Equal triangles having equal altitudes are on equal bases.*

3. ABC, DEF and GHK are equal triangles on equal bases BC, EF , and HK in the same straight line BK , and on the same side of it. Show that A, D , and G are in the same straight line.

4. If a quadrilateral is divided into two equal parts by one of its diagonals, then this diagonal bisects the other.

5. $ABCD, EBCF$ are equal trapezoids on the same base BC , and between the same $\parallel^s AF, BC$. Show that $AD=EF$.

6. *A median of a triangle bisects every straight line parallel to the side bisected by that median, and intercepted by the other sides.*

PROPOSITION 41. THEOREM.

If a parallelogram and a triangle are on the same base and between the same parallels, the parallelogram is double of the triangle.

(Proof given on page 143.)

1. *If a parallelogram and a triangle are on equal bases and between the same parallels, the parallelogram is double of the triangle.*

2. $KLMN$ is a \parallel^m of which the sides KN and LM are produced. If any point P be taken between the prolongations of these sides, then the area of the triangle PKL = the area of the quadrilateral $PNLM$.

3. The st. lines drawn through the vertices of a quadrilateral \parallel to its diagonals form a \parallel^m which is double of the quadrilateral.

4. The middle points of the sides of a quadrilateral are the vertices of a \parallel^m whose area is half that of the quadrilateral.

PROPOSITION 42. PROBLEM.

To construct a parallelogram having its area equal to that of a given triangle, and one of its angles equal to a given angle.

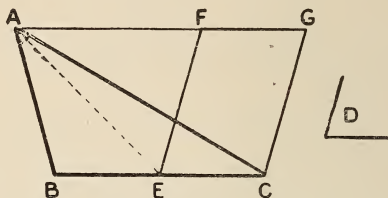


FIG. 196.

Let $\triangle ABC$ be the given \triangle , and D the given \angle .

It is required to construct a \parallel^m having its area equal to that of $\triangle ABC$, and one of its angles equal to $\angle D$.

Const. Bisect BC at E .

I.10.

At E make $\angle CEF = \angle D$.

I.23.

Through A draw $AG \parallel$ to BC , and through C draw $CG \parallel$ to EF .

I.31.

Then $FECG$ is the \parallel^m required.

Join AE .

Proof. Now $\triangle^s AEB, AEC$ are on equal bases and between the same \parallel^s ;

$\therefore \triangle AEB = \triangle AEC$.

I.38.

Hence $\triangle ABC$ is double of $\triangle AEC$.

But $FECG$ is a \parallel^m by construction,

Def.55.

and it is double of $\triangle AEC$, for they are on the same base and between the same \parallel^s .

I.41.

$\therefore \parallel^m FECG = \triangle ABC$,

and it has one of its $\angle^s CEF = \angle D$.

Q.E.F.

1. Construct a triangle having its area equal to that of a given parallelogram, and an angle equal to a given angle.
2. Construct a parallelogram whose area and perimeter shall be equal to those of a given triangle.
3. On the side of a given square construct an equivalent parallelogram, having an angle equal to a given angle.

PROPOSITION 43. PROBLEM.

*The complements of the parallelograms about the diagonal of a parallelogram, are equal in area.**

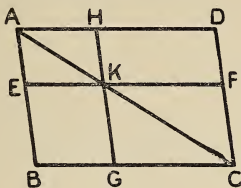


FIG. 197.

Let $ABCD$ be a \parallel^m , and KB , KD the complements of the \parallel^{ms} EH , GF about the diagonal AC .

It is required to prove that the area of the complement KB is equal to that of the complement KD .

Proof. Because AK is a diagonal of \parallel^m EH ,

$$\therefore \triangle AEK = \triangle AHK. \quad I.34.$$

Similarly $\triangle KGC = \triangle KFC$;

$$\therefore \text{sum of } \triangle^s AEK, KGC = \text{sum of } \triangle^s AHK, KFC.$$

But the whole $\triangle ABC =$ the whole $\triangle ADC$; I.34.

\therefore taking away equals from equals, the remainder, the complement $KB =$ the remainder, the complement KD . Q.E.D.

* If through any point in the diagonal of a parallelogram straight lines are drawn parallel to its sides, the parallelogram will be divided by them into four parallelograms. Of these, the two through which the diagonal passes are said to be *about the diagonal*, and the other two, which with them make up the whole parallelogram, are said to be their *complements*.

1. In the figure to Prop. 43 prove that

(i) $\parallel^m BF = \parallel^m HC$.

(ii) $\triangle KAB = \triangle KAD$.

(iii) EH is \parallel to GF .

2. The \parallel^m s about a diagonal of a square are squares.

3. The \parallel^m s about a diagonal of a rhombus are rhombuses.

4. Through the point P within the parallelogram $KLMN$ st. lines are drawn \parallel to its sides. If the \parallel^m s PL , PN are equal, then P is in the diagonal KM .

PROPOSITION 44. THEOREM.

On a given base to construct a parallelogram having its area equal to that of a given triangle, and one of its angles equal to a given angle.

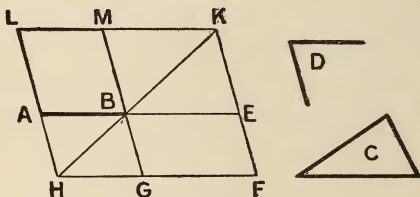


FIG. 198.

Let AB be the given st. line, C the given \triangle , and D the given \angle .

It is required to construct on AB a \parallel^m having its area equal to that of $\triangle C$, and one of its angles equal to $\angle D$.

Const. On AB produced construct $\parallel^m BEFG = \triangle C$, and having $\angle EBG = \angle D$. I.22, I.42.

Produce FE , FG , GB .

Through **A** draw **HAL** \parallel to **BG** or **EF**, meeting **FG** produced at **H**. I.31.

Join **HB**, and produce it to meet **FE** produced at **K**.

Through **K** draw **KML** \parallel to **EA** or **FH**, meeting **GB** produced at **M**, and **HL** at **L**. I.31.

Then **LABM** is the \parallel^m required.

Proof. Because **HF** meets the \parallel^s **HA**, **FE**,

\therefore sum of \angle^s **AHF**, **HFE** = two rt. \angle^s . I.29.

\therefore sum of \angle^s **BHF**, **HFE** is less than two rt. \angle^s ;

hence **HB**, **FE** meet when produced towards **B** and **E**, as in the construction. I.29. Cor.

Similarly **HA** and **GB** when produced both meet **KL**.

\therefore **LHFK** and **LABM** are both \parallel^{ms} .

Again because **LHFK** is a \parallel^m , of which **HK** is a diagonal, and **AG**, **ME** are \parallel^{ms} about **HK**,

\therefore complement **BL** = complement **BF**. I.43.

But $\triangle C = BF$; Const.

\therefore **BL** = $\triangle C$.

Also, $\angle ABM = \angle EBG$. I.15.

But $\angle D = \angle EBG$; Const.

\therefore $\angle ABM = \angle D$. Q.E.F.

1. On a given base to construct a rectangle equal in area to a given rectangle.

2. On the base of a given triangle to construct a \parallel^m equal to the triangle, and having an angle equal to a given angle.

3. On the base of a given triangle to construct a rectangle equal to the triangle.

4. On a given base to construct an isosceles \triangle equal to a given \parallel^m .

PROPOSITION 45. PROBLEM.

To construct a parallelogram having its area equal to that of a given rectilineal figure, and one of its angles equal to a given angle.

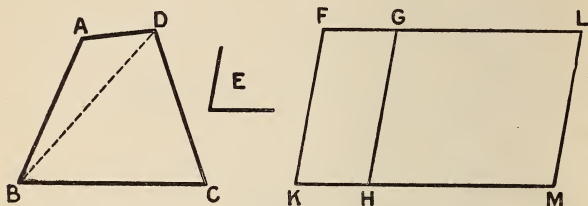


FIG. 199.

Let $ABCD$ be the given rectilineal figure, and E the given \angle .

It is required to construct a \parallel^m having its area equal to $ABCD$, and one of its angles equal to $\angle E$.

Const. Join BD . Construct $\parallel^m FH = \triangle ABD$, and having $\angle FKH = \angle E$. I.42.

On GH construct $\parallel^m GM = \triangle BCD$, and having $\angle GHM = \angle E$. I.44.

Then $FKML$ is the \parallel^m required.

Outline of Proof. (i) KH , HM are in the same st. line. (ii) FG , GL are in the same st. line. (iii) KF , ML are \parallel , and $FKML$ is a \parallel^m . (iv) $\parallel^m FKML =$ rectilineal figure $ABCD$, and $\angle FKM = \angle E$.

Write the proof in full.

1. To construct a rectangle equal to a given hexagon.
2. To construct a rectangle equal to the sum of two given parallelograms.
3. To construct a rectangle equal to the difference of two given rectangles.
4. On a given base construct a rectangle equal to a given quadrilateral.

5. On a given base construct a rectangle equal to the sum of two given parallelograms.

6. On a given base construct a rectangle equal to the difference of a given triangle and a given parallelogram.

PROPOSITION 46. PROBLEM.

To construct a square on a given straight line.

Let AB be the given st. line: *it is required to construct a square on AB .*

Const. From A draw AC at rt. \angle^s to AB ,
and from AC cut off $AD = AB$.

I.11.

I.3.

Through D draw $DE \parallel$ to AB , and
through B draw $BE \parallel$ to AD , meeting DE
at E .

I.31.

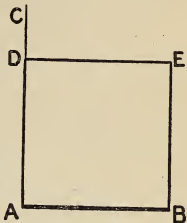


FIG. 0

Then $ABED$ is the square required.

Proof. Because DE is \parallel to AB , and BE is \parallel to AD ;

$\therefore ABED$ is a \parallel^m .

Def.55

Also, $\angle BAD$ is a rt. \angle ;

Const.

\therefore the $\parallel^m ABED$ is a rectangle.

Def.56.

Also, the adjacent sides AB, AD are equal;

Const.

\therefore the rectangle $ABED$ is a square,

Def.58.

and it is constructed on the given st. line AB .

Q.E.F.

1. If the diagonals of a \parallel^m are equal and at right angles to each other, the \parallel^m is a square.

2. To construct a square when the sum of a side and a diagonal are given.

3. *The squares on equal straight lines are equal; and, conversely, equal squares are on equal straight lines.*

PROPOSITION 47. THEOREM.

*In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.**

(Proof given on pp. 144-145.)

1. *In a right-angled triangle the difference between the square on the hypotenuse and the square on either side, is equal to the square on the other side.*

2. *If the diagonals of a quadrilateral are \perp to each other, the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair.*

3. *The square on the diagonal of a given square is equal to twice the given square.*

4. *The sum of the squares on the sides of a rectangle is equal to the sum of the squares on its diagonals.*

5. *ABC is an isosceles right-angled \triangle of which BAC is the rt. \angle . AD is the \perp drawn from A to BC. Show by means of this construction that the square on any given straight line is equal to four times the square on half the line.*

6. *ABC is a \triangle , and AD is the \perp drawn from A to BC. Show that diff. of sqq. on BD, DC = diff. of sqq. on AB, AC.*

7. *Construct a square equal to (i) the sum of two given squares, (ii) the difference of two given squares.*

8. *Construct a square equal to the sum of the squares on AB, CD and EF.*

9. *Divide a given straight line into two parts so that the sum of the squares on the parts may be equal to (i) the square on a given straight line, (ii) the sum of two given squares.*

10. *Divide a given straight line into two parts so that the square on one part may be equal to twice the square on the other.*

11. *Prove I. 47, describing the squares on the sides of the given triangle towards the triangle instead of away from it. Show that the figure to I. 47 admits of eight different forms.*

*The discovery of this important truth is generally ascribed to Pythagoras (580-510 B.C.). According to Proclus, Pythagoras gave to Geometry the form of a deductive science, by showing the dependence of geometrical truths on first principles.

PROPOSITION 48. THEOREM.

If the square on one side of a triangle is equal to the sum of the squares on the other two sides, then the angle contained by these two sides is a right angle.

Let the sq. on BC , a side of $\triangle ABC$, be equal to the sum of the sqq. on AB , AC : it is required to prove that $\angle BAC$ is a rt. \angle .

Const. From A draw AD at right angles to AC ,
and make $AD = AB$.

I.11.

I.3.

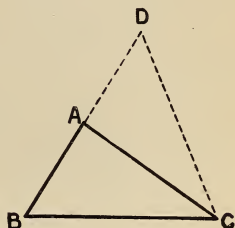


FIG. 201.

Join DC .

Proof. Because $AD = AB$,
 \therefore sq. on $AD =$ sq. on AB .

Const.

To each of these equals add sq. on AC ;
 \therefore sum of sqq. on AD , AC , = sum of sqq. AB , AC .

Again, because $\angle DAC$ is a rt. \angle ,
 \therefore sq. on $DC =$ sum of sqq. on AD , AC .

Const.

I.47.

But sq. on $BC =$ sum of sqq. on AB , AC .
 \therefore sq. on $DC =$ sq. on BC .

Hyp.

$\therefore DC = BC$.

Then in $\triangle^s ADC$, ABC , because

$$\left\{ \begin{array}{l} AD = AB, \\ AC = AC, \\ \text{and } DC = BC; \end{array} \right.$$

Const.

Proved.

$\therefore \angle DAC = \angle BAC$.

I.8.

But $\angle DAC$ is a right \angle by construction.

$\therefore \angle BAC$ is a rt. \angle .

Q.E.D.

1. The angle opposite to any side of a triangle is greater than, equal to, or less than a right angle, according as the square on that side is greater than, equal to, or less than the sum of the squares on the other two sides.

2. In the triangle ABC the square on BC is equal to the difference of the squares on AB , AC . Show that ABC is a right-angled triangle.

3. If the sum of the sqq. on one pair of opposite sides of a quadrilateral is equal to the sum of the sqq. on the other pair the diagonals of the quadrilateral are perpendicular to each other.

4. Let X be a point without the rectangle $KLMN$ such that the sum of the sqq. on the \perp^s from X on the four sides of the rectangle is equal to the sq. on KM . Show that the \angle^s KXM , LXN are rt. \angle^s .

Loci.

1. The st. line OP rotates about the fixed point O . Find the locus of its middle point when P moves in st. line.

2. The area of the triangle PAB is constant. Find the locus of P , the points A , B being fixed.

3. A , B are two fixed points. The point X moves so that the sum of the sqq. on AX , BX is always equal to the sq. on AB . Find the locus of X .

4. Find the locus of the points at which the diagonals of equal parallelograms on a fixed base intersect each other.

5. If st. lines are drawn parallel to the sides of a triangle from any point in its base, show that the diagonals of every parallelogram thus formed intersect in a st. line parallel to the base.

6. Construct a triangle when the base, the median bisecting the base, and the area are given.

7. Find the locus of the middle points of the portions of parallel st. lines intercepted by two intersecting transversals.

EXERCISES.

1. ABC is an equilateral triangle. P , Q and R are points in AB , BC and CA such that $AP=BQ=CR$. Show that PQR is an equilateral triangle.

2. In the triangle LMN the angle M is acute. Find a point O in ML or ML produced so that $MO=NO$.

3. Two circles whose centres are P and Q intersect in L and M . Show that the triangles PLQ and PMQ are identically equal.

4. If the sides of a quadrilateral are equal to one another, each of the diagonals bisects the angles through which it passes.

5. If a quadrilateral has all its sides equal, and also its diagonals equal, then all its angles are equal.

6. The diagonals of a rhombus bisect each other at right angles.

7. The diagonals of a kite are perpendicular to each other, and one of them bisects the other.

[A kite is a quadrilateral having two pairs of adjacent sides equal.]

8. Construct a rhombus of which two vertices shall be the given points A and B . Are the given conditions sufficient to determine the figure? Give reasons for answer.

9. Construct a rhombus when the diagonals are given.

10. Construct a rhombus having a side and a diagonal respectively equal to the given straight lines AB and CD .

11. In the figure to I. 1 let AB be produced both ways to meet the circumferences of the circles at X , Y . Show that CXY is an isosceles triangle.

12. To construct an equilateral triangle when the perimeter is given.

13. To construct an equilateral triangle when the perpendicular from the vertex to the base is given.

14. On the same base and on the same side of it there cannot be more than one equilateral triangle.

15. The sum of the distances of any point from the vertices of a triangle is greater than half the sum of the sides of the triangle.

16. Through a given point draw a straight line making equal angles with two intersecting straight lines.

17. Construct a triangle when the base, an angle at the base, and the difference of the other two sides are given.

18. Construct a triangle when one angle, the side opposite to it, and one of the other sides are given.

19. Construct a triangle when two sides and an angle opposite to one of them are given. Show that two different triangles can be constructed according to these conditions.

20. ABC and DBC are two triangles on the same base BC , and on the same side of it. If AB and DB are equal then AC and DC are unequal.

21. BCD is a circle whose centre is A . From E any point in AB let EF be drawn at right angles to AB , meeting the circumference in F , and from AF let AG be cut off equal to AE . Show that BG is perpendicular to AF .

22. $PQRS$ is a quadrilateral whose opposite sides are equal, and X is a point within it. If $PX = RX$ and $QX = SX$, show that PXR and QXS are straight lines.

23. The point at which the bisectors of the angles at the base of an isosceles triangle intersect is equidistant from the extremities of the base.

24. In a straight line of unlimited length fixed in position find a point equidistant from two fixed points. When is a solution impossible?

25. Find a point equidistant from three given points. When is a solution impossible?

26. Show that any point not in the \perp bisector of a straight line is unequally distant from the extremities of that line.

27. In the figure to Prop. 5 show that if BC be produced both ways to X and Y so that BX may be equal to CY , then AXY is an isosceles triangle.

28. The straight lines joining the middle points of the sides of an isosceles triangle form an isosceles triangle.

29. The straight line drawn through the vertices of two isosceles triangles, having a common base, bisects both vertical angles, and is the perpendicular bisector of the base.

30. If the bisector of the vertical angle of a triangle bisects the base also, the triangle is isosceles.

31. If any point X be taken in the base QR of the isosceles triangle PQR , then PX is less than PQ .

32. In the triangles ABC , DBC , on the same side of the base BC , the sides AB , AC , are respectively equal to the sides DC , DB . If AC and DB intersect at X , show that triangle ADX is isosceles.

33. If the vertical angle of a triangle is equal to the sum of the angles at the base, then the base is equal to twice the median bisecting it.

34. Place a straight line of given length between two intersecting straight lines so as to form equal angles with them.

35. From a given point draw two straight lines forming equal angles with two given straight lines which intersect each other.

36. In a given straight line find a point such that the perpendiculars drawn from it to two given straight lines may be equal.

37. Any two exterior angles at the base of an isosceles triangle are together greater than two right angles by the vertical angle.

38. The angle formed by the bisectors of the angles at the base of an isosceles triangle is equal to one of the exterior angles at the base.

39. If the bisector of an exterior angle of a triangle is parallel to one of the sides the triangle is isosceles. Is the converse of this true?

40. LMN is an isosceles triangle. At O , a point in the base MN , a perpendicular is erected cutting ML at P and NL produced at Q . Show that LPQ is an isosceles triangle.

41. The perpendiculars from the extremities of the base of an isosceles triangle to the opposite sides are equal. Prove the converse of this theorem.

42. The sum of the perpendiculars drawn from any point in the base of an isosceles triangle to the opposite sides is equal to the perpendicular drawn from either extremity of the base to the opposite side.

43. On the sides of any triangle, ABC , equilateral triangles BCD , CAE , ABF are described, all external; show that the straight lines AD , BE , CF are all equal.

44. State Euclid's axiom of superposition. What assumption is implied in it, but not expressed? What different sets of given conditions enable us to prove directly by this axiom the congruence of two triangles? Illustrate by means of diagrams.

45. If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angles supplementary, the triangles are equal in area.

46. Construct a right-angled triangle having given the hypotenuse and the difference of the sides.

47. Construct a triangle of given area having two of its sides respectively equal to two given st. lines.

48. One of the acute angles of a right-angled triangle is three times as great as the other ; trisect the smaller of these angles.

49. How is the locus of a point equally distant from two intersecting straight lines affected by the rotation of one of the lines about the point of intersection ? Discuss the case in which the angle formed by the lines vanishes.

50. How is the locus of a point equally distant from two parallel straight lines affected by the motion of one of the lines in such a way that it is always parallel to its first position ?

51. Find the locus of the point \bar{P} when it is always the vertex of an isosceles triangle on the given base AB . How is the vertical angle affected as P moves away from the base ?

52. Find the locus of a point whose distance from the given point P is equal to half its distance from the given point Q .

53. In the triangle ABC the side AB is greater than the side AC . If D is the middle point of BC , show :

(i) That any point in DA , or DA produced towards A , is farther from B than from C .

(ii) That any point in AD produced towards D is nearer to B than to C .

54. The medians of a triangle divide one another in the ratio of $1 : 2$.

55. Construct a triangle, two sides and the median bisecting the third side being given. .

56. If two medians of a triangle are equal the triangle is isosceles. State the converse of this theorem, and prove it.

57. Any two medians of a triangle are together greater than the third.

58. The three medians of a triangle are together less than the perimeter.

59. The perpendiculars to the sides of a triangle from the opposite vertices are concurrent.

60. The three triangles formed by the bisectors of the exterior angles of a triangle on its sides are equiangular.

61. The bisectors of the three interior and three exterior angles of a triangle meet three by three at four points.

62. $ABCD$ is a square of which the side AD is produced to E . Show that BE is greater than AC .

63. A diameter of a circle is greater than any other straight line in the circle which is not a diameter.

64. Show that it is always possible to construct a triangle whose sides are equal to three given straight lines, respectively, if any two of these lines are together greater than the third.

65. Construct a triangle, having given its perimeter and the angles at its base.

66. ABC is an isosceles triangle; find points D and E in the equal sides AB , AC , such that BD , DE , EC shall be all equal.

67. In a given straight line to find a point such that the difference of its distances from two fixed points shall be the greatest possible.

68. If from the vertical angle of a triangle three straight lines be drawn to the base, the first bisecting the angle, the second bisecting the base, and the third perpendicular to the base, the first is intermediate in length and position to the other two.

69. The sum of the perpendiculars to the sides of an equilateral triangle from any point within it is constant.

70. In one side of a triangle find a point such that the difference between the perpendiculars from it on the other two sides may be equal to a given straight line.

71. (a) The sum of the distances of any point in the base of an isosceles triangle from the two other sides is constant.

(b) What will the result be if the point is taken in the base produced?

72. Find a point such that the perpendiculars from it on two given straight lines shall be equal to two given straight lines, respectively.

73. If the opposite sides of a hexagon are equal and parallel, three of its diagonals are concurrent.

74. In the figure to I. 1, if AB is produced both ways to meet the circles in D and E , and from C , CD and CE are drawn, show that CDE is an isosceles triangle having each of the angles at the base, equal to one-fourth of the vertical angle.

75. To construct a triangle when the base, the difference of the sides and the difference of the angles at the base are given.

76. To construct a triangle whose medians shall be respectively equal to three given straight lines.

77. Find the locus of the point X when the area of the $\triangle ABX$ is constant, A , B , being fixed points.

78. PR and QS are parallel radii of two given circles whose centres are P and Q respectively. The straight line RS cuts the circumferences of these circles again at X and Y ; then PX is parallel to QY .

79. To draw a straight line equal and parallel to a given straight line and having its extremities in two given straight lines.

80. Between two given straight lines draw a straight line equal to one given straight line, and parallel to another.

81. Construct a rhombus equal in area to any given quadrilateral.

82. Construct a parallelogram equal in area and perimeter to a given triangle.

83. Construct a parallelogram when two diagonals and side are given.

84. Find the locus of the vertices of triangles on the same base and of equal altitude.

85. Find the locus of the point at which the diagonals of \parallel^{ms} on the same base and between the same \parallel^{s} intersect.

86. Show that when two angles have their arms parallel, respectively, their bisectors are either parallel or perpendicular.

87. ABC is a triangle. AC is produced to D and BC to E so that CD is equal to AC and CE to BC . Show that DE is parallel to AB .

88. Every st. line drawn through the points of intersection of the diagonals of a \parallel^{m} bisects the \parallel^{m} .

89. Through a given point P draw a st. line PXY meeting two given parallel st. lines in X and Y , such that XY may be equal to a given st. line.

90. P and Q are points in the \parallel st. lines AB , CD . PQ is bisected at X . Show that all st. lines drawn through X and terminated in AB , CD are bisected at X .

91. Construct a \parallel^{m} when one of the diagonals, the sum of two adjacent sides, and the angle between these sides are given.

92. If the vertices of one \parallel^{m} lie on the sides of another, the diagonals of the \parallel^{ms} are concurrent.

93. Three given st. lines are drawn from a point : draw another st. line cutting them so that the two segments of it intercepted by the given lines may be equal.

94. A point P moves along the circumference of a circle from one extremity A of the diameter AB to the extremity B . If O is a point in AB nearer to A than to B , show that OP increases throughout the motion.

95. If the diagonals of a parallelogram are equal it is a rectangle ; and if they also cut at right angles it is a square.

96. To bisect a quadrilateral by a straight line drawn through (a) one of its vertices ; (b) a given point in one of its sides.

97. $ABCD$ is a \parallel^m . AD , BC are bisected at E , F . Show that BE , DF intercept one-third of AC .

98. On the sides AB , AC of a triangle ABC , parallelograms $ABDE$, $ACFG$ are described, and DE , FG are produced to meet in H . Show that the area of these two parallelograms together is equal the area of a parallelogram having BC for one of its sides and for the adjacent side a line equal and parallel to AH .

99. $ABCD$ is a parallelogram ; from D any straight line DFG is drawn meeting BC at F and AB produced at G ; draw AF and CG . Show that the triangle AFB is equal to the triangle CFG .

100. $PQRS$ is a parallelogram. PQX and PSY are equilateral triangles external to the parallelogram. Show that RX , RY , and XY are equal to one another.

101. In a side of a triangle find the point from which the straight lines drawn parallel to the other sides of the triangle and terminated by them are equal.

102. ABC is an isosceles triangle, of which A is the vertex ; AB , AC are bisected in D and E respectively ; BE , CD intersect in F : show that the triangle ADE is equal to three times the triangle DEF .

103. If a straight line intercepted between one extremity of the base of an isosceles triangle, and the opposite side (produced if necessary) is equal to a side of the triangle, the angle formed by this line and the base produced, is equal to three times either of the angles at the base of the triangle.

104. If three straight lines are drawn from the base to the opposite sides of an isosceles triangle, making equal angles with the base, viz., one from its extremity, the other two from any other point in it, these two are together equal to the first.

105. PQRS is a quadrilateral of which PS is the greatest side and QR the least. Prove $\angle PQR$ is greater than $\angle PSR$, and $\angle QRS$ is greater than $\angle QPS$.

106. ABCD is a quadrilateral having AB equal to AD but BC greater than CD. AC and BD intersect at O. Prove that OB is greater than OD.

107. Determine the limits of the sum of the areas of the complements of the \parallel^m s about the diagonal of a given \parallel^m .

108. The bisectors of the four interior angles of a \parallel^m enclose a rectangle whose diagonals are each \parallel to a pair of opposite sides of the given \parallel^m . Discuss the cases in which the area of the rectangle thus formed vanishes.

109. Within a quadrilateral find a point such that the sum of the distances from it to the vertices shall be the least possible.

110. In any trapezium the straight line joining the middle points of the two sides which are not parallel is parallel to the other two sides and equal to half their sum; and the diagonals of the trapezium intercept on that line a portion equal to half the difference of the two parallel sides.

111. Construct a trapezium whose sides are respectively equal to four given straight lines. Can more than one trapezium be constructed according to the data here given?

112. In any pentagon if a side at each of three corners be produced, the sum of the three exterior angles thus formed is equal to the sum of the interior angles at the other two corners.

113. ABC is a right-angled triangle of which AC is the hypotenuse. In AC find a point X such that AX may be equal to the \perp drawn from X to AB.

114. The perimeter of an isosceles triangle is greater than that of a rectangle of equal area and altitude.

115. Divide a straight line into two parts such that the square on one of them may be double the square on the other.

116. In a straight line AB, or AB produced, find a point D such that the difference of the squares on AD, BD may be equal to a given square.

117. Trisect a triangle by straight lines drawn from a given point in one of its sides.

118. Given two sides of a triangle and the straight line drawn from the extremity of one to the middle of the other; construct the triangle.

119. If the lengths of the sides of a parallelogram be given, show that its area will be greatest when it is a rectangle.

120. The perimeter of an isosceles triangle is less than that of any other triangle equal to it on the same base.

121. Of all the squares whose vertices may lie on the sides of a given square, which one has the least area? Construct it.

122. ABC is a triangle right-angled at A ; K is the corner of the square on AB opposite to A , and H is the corner of the square on AC opposite to A ; AB is produced to D so that AD is equal to CK , and AC is produced to E so that AE is equal to BH ; then shall CD be equal to BE .

123. In an acute-angled triangle the square on any side is less than the sum of the squares on the other two sides.

124. In an obtuse-angled triangle the square on the side opposite to the obtuse angle is greater than the sum of the squares on the other two sides.

125. $ABCD$ is a rectangle, and X is any point in the same plane. Show that the sum of the squares on AX , CX is equal to the sum of the squares on BX , DX .

126. PQR is a triangle of which the angle PQR is a right angle. PQ , QR are bisected at X , Y . Show that five times the square on PR is equal to four times the sum of the squares on PY , RX .

127. The sides of a parallelogram are the hypotenuses of four right-angled isosceles triangles which are external to the parallelogram. Show that their vertices remote from the parallelogram are the vertices of a square.

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